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Regular and Orientably-Regular Maps with Quasiprimitive Automorphism Groups on Vertices

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Abstract

Regular and orientably-regular maps are central to the part of topological graph theory concerned with highly symmetric graph embeddings. Classification of such maps often relies on factoring out a normal subgroup of automorphisms acting imprimitively on the set of the vertices of the map. Maps whose automorphism groups act quasiprimitively on their vertices do not allow for such factorization. Instead, we rely on classification of quasiprimitive group actions which divides such actions into eight types, and we show that four of these types, **HS**, **HC**, **SD**, and **CD**, do not occur as the automorphism groups of regular or orientably-regular maps. We classify regular and orientably-regular maps with automorphism groups of the **HA** type, and construct new families of regular as well as both chiral and reflexible orientably-regular maps with automorphism groups of the **TW** and **PA** types.

Introduction

Maps are 2-cell embeddings of graphs in compact, connected surfaces. Regular maps have the largest automorphism groups possible, acting regularly on flags (edges with a

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longitudinal and a transverse direction) of the map. Similarly, orientably-regular maps are maps in orientable surfaces that have the largest possible orientation preserving automorphism groups that act regularly on darts (edges with longitudinal direction). Regular and orientably-regular maps constitute the most meaningful generalization of the Platonic solids. Early recognition of the importance of regular maps in modern mathematics goes back to Kepler [15]; more recent development of the theory of maps was closely related to the theory of map colorings, with the topic of highly symmetric maps always at the center of interest. The study of regular maps is nowadays considered one of the 'classical' areas of mathematics (e.g., Heffter [10], Klein [16], Dyck [8], or Burnside [3]).

The aim of our paper is to contribute to the classification of finite regular and orientably-regular maps by looking at their 'irreducible quotients'. The starting point is the observation that if K is a normal subgroup of the automorphism group of a regular map $\mathcal{M}$ which acts imprimitively on the vertex set of $\mathcal{M}$, then $\mathcal{M}$ can be viewed as a lift of the factor map $\mathcal{M}/K$ whose vertices are the blocks of imprimitivity of K. Consequently, a regular map $\mathcal{M}$ with the property that each non-trivial normal subgroup of its automorphism group acts transitively on vertices can be viewed as *irreducible* with regard to this factoring process, and as such, needs to be studied on its own. The same considerations apply, of course, to orientably-regular maps. Permutation groups in which every non-trivial normal subgroup is transitive are known as *quasiprimitive*. Summing up, a study of regular (or orientably-regular) maps whose full (or orientation preserving) automorphism groups act quasiprimitively on their vertices can be viewed as the study of the irreducible building stones for the entire class of (orientably-)regular maps.

Quasiprimitive permutation groups have been characterized and shown to belong to eight distinct types in [26]; denoted therein by the acronyms **HA**, **HS**, **HC**, **AS**, **SD**, **CD**, **TW**, and **PA**. In the present article, we show that four of these classes, **HS**, **HC**, **SD**, and **CD**, do not occur as the full automorphism groups of regular or orientably-regular maps. We fully classify the regular and orientably-regular maps whose automorphism groups belong to the **HA** type (in the orientably-regular case each time determining whether the resulting map is chiral or reflexible), and we construct previously unknown infinite families of regular and orientably-regular maps whose automorphism groups belong to the **TW** and **PA** types. The last, **AS**, type consisting of almost simple groups is only briefly summarized in our article as this class has been extensively studied before, and the classification of (orientably-)regular maps whose automorphism groups are of this type appears to be far from anybody's reach at this point.

1 Fundamental concepts

A group G acting on a set X is said to act *regularly*, if for any pair of elements $x, y \in X$ there exists a unique element $g \in G$ mapping x to y, $x^g = y$. In such a case, X can be identified with the elements of G, and consequently, any mathematical structure with an automorphism group acting regularly on its base set can be identified with the group itself, the building blocks of the structure being identified with cosets of stabilizers of some fixed blocks. This identification has been used in the theory of regular and orientably-regular maps as well and we just sum up the basics, referring for details to [14] and [2] for the theory of maps on orientable and on general surfaces, and to [29] for a recent survey of the theory of regular and orientably-regular maps. In all the forthcoming group presentations we will assume that the listed exponents are the *true orders* of the corresponding elements.

A finite regular map $\mathcal{M}$ can in this way be identified with a (partial) three-generator presentation of a finite group G, isomorphic to the automorphism group $\mathrm{Aut}(\mathcal{M})$ of $\mathcal{M}$, of the form

$$G = \langle x, y, z \mid x^2, y^2, z^2, (xy)^2, (yz)^\ell, (zx)^m, \ldots \rangle \tag{1}$$

where dots indicate possible presence of additional relators (at least one if the carrier surface of the map is not simply connected). In particular, all vertices of $\mathcal{M}$ have degree ℓ and all the face boundary walks in $\mathcal{M}$ have length m; we will often refer just to *face length* m. The pair (ℓ, m) is the *type* of our regular map $\mathcal{M}$. In such a representation of $\mathcal{M}$, its flags are elements of G, its darts are (say) right cosets of the subgroup $\langle x\rangle$, while edges, vertices and faces are right cosets of the dihedral subgroups $\langle x, y\rangle$, $\langle y, z\rangle$ and $\langle z, x\rangle$ of order 4, ℓ and m, respectively; their mutual incidence is given by non-empty intersection. The group G then acts on the above cosets as the automorphism group of $\mathcal{M}$ simply by right multiplication. Readers acquainted with maps will recognize that the three generators x, y, z correspond to involutory automorphisms of $\mathcal{M}$ taking a fixed flag onto its three neighboring flags, and the three dihedral subgroups correspond to edge-, vertex- and a face-stabilizers of $\mathcal{M}$. We will write $\mathcal{M} = \mathrm{Map}(G; x, y, z)$ to formally identify a regular map $\mathcal{M}$ with a group presentation as in (1).

The algebraic situation with finite orientably-regular maps is similar. Each such map $\mathcal{M}$ can be identified with a partial two-generator presentation of a group H, isomorphic to the group $\mathrm{Aut}^+(\mathcal{M})$ of orientation-preserving automorphisms of $\mathcal{M}$, of the form

$$H = \langle \lambda, \rho \mid \lambda^2, \rho^\ell, (\lambda\rho)^m, \ldots \rangle\ . \tag{2}$$

Here, elements of H represent darts of $\mathcal{M}$, right cosets of the cyclic groups $\langle\lambda\rangle$, $\langle\rho\rangle$ and $\langle\lambda\rho\rangle$ represent edges, vertices and faces of $\mathcal{M}$, and H acts on the cosets, and hence on the map, as the group of orientation-preserving automorphisms via right multiplication. The generators λ and ρ, stabilizing an edge e and a vertex v incident to e, represent a half-turn of $\mathcal{M}$ about the center of e and a $2\pi/\ell$ turn of $\mathcal{M}$ about v in accord with a chosen orientation of the carrier surface of the map. Again, the pair (ℓ, m) is the type of the map, and we will use the notation $\mathcal{M} = \mathrm{Map}(H; \lambda, \rho)$ in this case.

If a regular map $\mathcal{M} = \mathrm{Map}(G; x, y, z)$ is orientable (meaning that its carrier surface is orientable), $\mathcal{M}$ is also orientably-regular, with $\mathrm{Aut}^+(\mathcal{M}) = \langle \rho, \lambda\rangle$ for $\lambda = xy$ and $\rho = yz$. In fact, a regular map $\mathrm{Map}(G; x, y, z)$ is orientable if and only if the subgroup $\langle xy, yz\rangle$ has index 2 in G. Reversing the line of thought, an orientably-regular map $\mathcal{M} = \mathrm{Map}(H; \lambda, \rho)$ may also be regular. This happens if and only if the map admits an orientation-reversing automorphism, which (see e.g. [29]) is equivalent to the existence of an automorphism of H that fixes λ and inverts ρ. In such a case we call the orientably-regular map $\mathcal{M}$ *reflexible*; otherwise, that is, when $H \cong \mathrm{Aut}^+(\mathcal{M}) = \mathrm{Aut}(\mathcal{M})$, the map is called *chiral*.

Let now $\mathcal{M}$ be a regular map $\mathrm{Map}(G; x, y, z)$ or an orientably-regular map $\mathrm{Map}(H; \lambda, \rho)$, and let K be a normal subgroup of G or H. To avoid trivialities, we assume that

$x, y, z, xy \notin K$ (for regular $\mathcal{M}$) or $\lambda \notin K$ (for orientably-regular $\mathcal{M}$). In either case, one may form the *quotient map* $\mathcal{M}/K$ to be the regular map $\mathrm{Map}(G/K; \bar{x}, \bar{y}, \bar{z})$ or the orientably-regular map $\mathrm{Map}(H/K; \bar{\lambda}, \bar{\rho})$, with the obvious usage of $\bar{x} = Kx$, $\bar{y} = Ky$, $\bar{z} = Kz$, $\bar{\lambda} = K\lambda$ and $\bar{\rho} = K\rho$. The corresponding induced presentations of the groups $\bar{G} = G/K$ and $\bar{H} = H/K$ then are

$$\bar{G} = \langle \bar{x}, \bar{y}, \bar{z} \mid \bar{x}^2, \bar{y}^2, \bar{z}^2, (\bar{x}\bar{y})^2, (\bar{y}\bar{z})^{\ell'}, (\bar{z}\bar{x})^{m'}, \ldots \rangle, \quad \bar{H} = \langle \bar{\lambda}, \bar{\rho} \mid \bar{\lambda}^2, \bar{\rho}^{\ell'}, (\bar{\lambda}\bar{\rho})^{m'}, \ldots \rangle \tag{3}$$

where ℓ' and m' are divisors of ℓ and m, respectively. If K is intransitive on the vertex set of $\mathcal{M}$, the quotient map $\mathcal{M}/K$ has at least two vertices. Since, topologically, the map $\mathcal{M}$ branch-covers the quotient map $\mathcal{M}/K$, one may view the (regular, or orientably-regular) map $\mathcal{M}$ as a *lift* of a *non-trivial* (regular, or orientably-regular) quotient; with the non-triviality referring to having at least two vertices. From this point of view, the cases to be examined separately arise when $\mathcal{M}$ does not admit a normal subgroup of automorphisms acting imprimitively on vertices of $\mathcal{M}$. This is precisely the case when the above groups G and H act *quasiprimitively on vertices*. We reiterate that investigation of the class of regular and orientably-regular (reflexible or chiral) maps with quasiprimitive automorphism groups is the main goal of this article.

We continue by a note on non-orientable maps, those with a non-orientable carrier surface. If $\mathcal{M} = \mathrm{Map}(G; x, y, z)$ is non-orientable and regular, or, equivalently, if $G = \langle x, y, z \rangle = \langle \lambda, \rho \rangle$ for $\rho = yz$ and $\lambda = zx$, then the map $\tilde{\mathcal{M}} = \mathrm{Map}(G; \lambda, \rho)$ is orientably-regular and forms a double-cover of $\mathcal{M}$. Here, G is the (full) automorphism group $\mathrm{Aut}\mathcal{M}$ of $\mathcal{M}$ and, at the same time, the group $\mathrm{Aut}^+\tilde{\mathcal{M}}$ of orientation-preserving automorphisms of the double cover $\tilde{\mathcal{M}}$. Moreover, $\tilde{\mathcal{M}}$ is also reflexible, an obvious reflection being the inner automorphism ϑ of G given by conjugation by z. For the (full) automorphism group $\tilde{G}$ of $\tilde{\mathcal{M}}$ we then have $\tilde{G} = \mathrm{Aut}(\tilde{\mathcal{M}}) \cong \mathrm{Aut}^+(\tilde{\mathcal{M}})\langle\vartheta\rangle \cong G \times \mathbb{Z}_2$, because the element $\vartheta z = z\vartheta$ commutes with both λ and ρ in $G\langle\vartheta\rangle$. Observe also that $\tilde{G}$ is *never* quasiprimitive on vertices of $\tilde{\mathcal{M}}$ since $\tilde{G}$ acts on the non-trivial partition of the vertex set induced by the action of the direct factor $\mathbb{Z}_2 = \langle \vartheta z \rangle$.

Finally, it turns out that many of the orientably-regular maps obtained in the forthcoming sections fall in the class of Cayley maps the theory of which (without regularity assumptions) was initiated in [27] and further developed e.g. in [11] and [5]. Here we will restrict ourselves to *orientably-regular Cayley maps* $\mathcal{M} = \mathrm{Map}(H; \lambda, \rho)$ distinguished by the property $H = J\langle\rho\rangle$ for some subgroup $J \le H$ such that $J \cap \langle\rho\rangle = 1$. Since vertices of $\mathcal{M}$ are right cosets of $\langle\rho\rangle$ in H, it follows that the subgroup J acts regularly on the vertex set of the map, making the underlying graph of $\mathcal{M}$ a Cayley graph $C(J, S)$ for some unit-free inverse-closed generating set S of J.

In the even more special instance when J is *normal* in H, i.e., when H is a semi-direct product $J \rtimes \langle\rho\rangle$, we speak about a *normal* (orientably-regular) Cayley map. In this case, conjugation by ρ induces an automorphism $\hat{\rho}$ of J and its restriction $\pi = \pi_{\hat{\rho}}$ to S is a cyclic permutation of S. We will use the notation $\mathrm{CM}(J, \pi)$ to denote such a Cayley map. It turns out that either all elements in S are involutions, or none of them is and then $s^{-1} = s\hat{\rho}^{\ell/2} = \rho^{-\ell/2}s\rho^{\ell/2}$ for every $s \in S$, where ℓ is the order of ρ (necessarily even in this case). Conversely, given a normal orientably-regular Cayley map $\mathcal{M} = \mathrm{CM}(J, \pi)$, the cyclic permutation π of S extends to an automorphism of J corresponding to the element

ρ as above, so that $\mathrm{Aut}^+(\mathcal{M}) \cong J\langle\rho\rangle$. Moreover, since we also know that $J\langle\rho\rangle = \langle\lambda, \rho\rangle$, the involution λ can be taken to be equal to an arbitrary element of S in the all-involutions case, or to $s\rho^{k/2}$ for an arbitrary $s \in S$ if no element in S is involutory.

As already stated, our aim is to look at (orientably-) regular maps whose (orientation-preserving) automorphism groups are quasiprimitive permutation groups in their action on vertices of the maps; we will refer to these as *quasiprimitive* (orientably-) regular maps. To begin with this task, we need to make sure that the automorphism groups of the maps we consider act *faithfully* in their induced actions on the vertex sets. By results of [18], the automorphism group G of a regular map (or the orientation-preserving automorphism group H of an orientably-regular map) acts faithfully on the vertex set of the map if and only if the vertex-stabilizer in G and H, respectively, has no subgroup (except for the trivial group) normal in G (or H). Fortunately, this condition will be satisfied in all forthcoming sections by all considered families of maps.

In what follows, we will use the classification of finite quasiprimitive permutation groups of [26] that divides such groups into the following eight types: holomorph-Abelian (**HA**), holomorph-simple (**HS**), holomorph-compound (**HC**), almost simple (**AS**), simple diagonal (**SD**), compound-diagonal (**CD**), twisted wreath product (**TW**), and product action (**PA**). As it turns out, four of these eight types cannot be automorphism groups of (orientably-) regular maps, namely, the types **HS**, **HC**, **SD** and **CD**, as we will explain next after briefly introducing these types. We denote by $\mathrm{soc}(L)$ the socle of a group L, that is, the subgroup generated by the union of all minimal normal subgroups of L.

Lemma 1.1 *The four types of quasiprimitive groups* **HS, HC, SD, CD** *can not be the automorphism groups of any orientably-regular or regular maps.*

Proof. A quasiprimitive group G of type **HS** acts on the set of elements of a finite non-Abelian simple group T satisfying the inclusions $T \rtimes \mathrm{Inn}(T) \le G \le T \rtimes \mathrm{Aut}(T)$, with point stabilizer G_1 satisfying $\mathrm{Inn}(T) \le G_1 \le \mathrm{Aut}(T)$. A quasiprimitive group G of type **HC** acts on a set of the form T^k for T as above and $k \ge 2$, with $T^k \rtimes \mathrm{Inn}(T^k) \le G \le T^k \rtimes \mathrm{Aut}(T^k)$ and $\mathrm{Inn}(T^k) \le G_1 \le \mathrm{Aut}(T^k)$. Further, a quasiprimitive group G of type **SD** acts on a set as in the type **HC** but this time with $T^k = \mathrm{soc}(L) \le G \le L$ for some group L, with point stabilizer D in T^k being a diagonal subgroup, i.e., $D = \{(t, t^{\varphi_2}, \ldots, t^{\varphi_k}) \mid t \in T\} \cong T$ for some $\varphi_i \in \mathrm{Aut}(T)$, $2 \le i \le k$. Finally, a quasiprimitive group G of type **CD** is assumed to preserve a product structure on Ω, that is $\Omega = \Delta^\ell$ for some $\ell \ge 2$. In this case, for T as above, $T^k \le G \le H \wr S_\ell \le \mathrm{Sym}(\Delta) \wr S_\ell$ for some quasiprimitive group $H \le \mathrm{Sym}(\Delta)$ of type **SD** with $\mathrm{soc}(H) = T^m$, where $k = \ell m$, $m \ge 2$, and with point stabilizer D' in T^k of the form $D' = \Pi_{i=1}^{\ell} D_i$, where each D_i is a diagonal subgroup of T^m.

These descriptions imply that in the above four types the point stabilizer is never cyclic or dihedral. However, vertex stabilizers of (orientably-) regular maps are always cyclic or dihedral. Thus, no (orientably-) regular map has (orientation-preserving) automorphism group of type **HS**, **HC**, **SD** or **CD** acting quasiprimitively on its vertices. □

In what follows we will consider quasiprimitive (orientably-) regular maps with automorphism groups of the other four types, that is, types **HA**, **TW**, **PA**, and **AS**.

2 Quasiprimitive regular maps of type HA

In this section we derive a full classification of regular or orientably-regular maps $\mathcal{M}$ whose full automorphism groups or full orientation preserving automorphism groups G act quasiprimitively on their vertex sets and belong to the **HA** (Holomorph Affine) type of quasiprimitive permutation groups. For a description of quasiprimitive groups of the holomorph affine type (**HA**) we follow [26]. Thus, we identify the vertex set of $\mathcal{M}$ with the elements of the Galois field F_{p^r}, p prime, and assume that G is a subgroup of the group of affine transformations of F_{p^r}, with the socle $N = \mathbb{Z}_p^r$; the group of translations. Note that, in this case, the action of G on F_{p^r} is necessarily primitive [26]. We divide our classification into two cases, considering separately orientable and non-orientable maps.

2.1 Orientably-regular maps of type HA

Let G be the full orientation preserving automorphism group of an orientably-regular map $\mathcal{M}$, that is a map embedded in an orientable surface for which the group of orientation preserving automorphisms is equal to G, and suppose that G has a quasiprimitive action on the vertices of $\mathcal{M}$ of the **HA** type. As argued above, the vertex set of $\mathcal{M}$ may then be identified with the additive group of the Galois field of order p^r for some prime number p and positive integer r, $F_{p^r}^+ = \mathbb{Z}_p^r$, on which the group $\mathbb{Z}_p^r$ acts as a regular group of orientation preserving map automorphisms. Thus, $\mathcal{M}$ is a Cayley map of the group $\mathbb{Z}_p^r$ (Theorem 2.2. [27] or Section 1), $\mathcal{M} \cong CM(\mathbb{Z}_p^r, \pi)$, $X = \{x_1, x_2, \ldots, x_\ell\}$ generates $\mathbb{Z}_p^r$ (hence $r \le \ell$), and $\mathrm{Aut}^+\mathcal{M} = \mathbb{Z}_p^r \cdot \langle\varphi\rangle$, where φ is the generator of the stabilizer of $\vec{\mathbf{0}} = (0, 0, \ldots, 0)$ in $\mathrm{Aut}^+\mathcal{M} \cong G$. Since G is also assumed to be a subgroup of the group of affine transformations of $\mathbb{Z}_p^r$ and φ fixes $\vec{\mathbf{0}}$, we deduce that $G = \mathbb{Z}_p^r \rtimes \langle A\rangle$, for some $A \in \mathrm{GL}(r, p)$ of order $\ell = |X|$. This also means that all such orientable maps are normal Cayley maps, and the cyclic permutation π of X is induced by the action of A on X, $\pi(x_i) = x_i A$. Each element in G can be denoted by a pair (A^i, α), $0 \le i \le \ell - 1$, and $\alpha \in \mathbb{Z}_p^r$. The product of two elements (A^i, α) and (A^j, β) is equal to $(A^{i+j}, \alpha A^j + \beta)$, with the exponents taken modulo ℓ, and the action of G on $\mathbb{Z}_p^r$ is the natural action $\beta^{(A^i,\alpha)} = \beta A^i + \alpha$.

The above discussion yields the following characterization (Theorem 2.2) of orientably-regular maps whose quasiprimitive orientation preserving automorphism groups are of type **HA**. Recall that the action of G on $\mathbb{Z}_p^r$ is primitive if and only if $\langle A\rangle$ is an irreducible subgroup of $\mathrm{GL}(r, p)$, i.e., it admits no non-trivial invariant subspaces. In terms of the parameters of the group, we have the following well-known Lemma 2.1.

Lemma 2.1 *The quasiprimitive group $G = \mathbb{Z}_p^r \rtimes \mathbb{Z}_\ell$ of type* **HA** *is primitive in its natural action on $\mathbb{Z}_p^r$ if and only if ℓ is a primitive divisor of $p^r - 1$, that is ℓ divides $p^r - 1$ but for each $n < r$ the number $p^n - 1$ cannot be divided by ℓ.*

Theorem 2.2 *Let $G = \mathbb{Z}_p^r \rtimes \mathbb{Z}_\ell$ be a quasiprimitive group of type* **HA***, where $\mathbb{Z}_\ell$ is generated by an irreducible $A \in \mathrm{GL}(r, p)$, and let $\mathcal{M} = CM(\mathbb{Z}_p^r, \pi)$ be a normal Cayley map* *whose orientation preserving automorphism group acts quasiprimitively on the vertices of $\mathcal{M}$. Then G can be the automorphism group of $\mathcal{M}$ if and only if ℓ is even when $p \geq 3$ and $\pi = (\alpha, \alpha A, \ldots, \alpha A^{\ell-1})$ for some non-zero vector $\alpha \in \mathbb{Z}_p^r$. Moreover,*

(1) *$\mathcal{M}$ is of type $(\ell, \frac{\ell}{2})$, when $p \geq 3$ and $\ell \equiv 2 \bmod 4$;*

(2) *$\mathcal{M}$ is of type (ℓ, ℓ) when $p \geq 3$ and $\ell \equiv 0 \bmod 4$;*

(3) *$\mathcal{M}$ is of type (ℓ, ℓ) when $p = 2$.*

Proof. First note that if $p \geq 3$ (and $\mathbb{Z}_p^r$ contains no involutions), the requirement that $\{\alpha, \alpha A, \ldots, \alpha A^{\ell-1}\}$ is closed under taking inverses is equivalent to requiring that ℓ is even. Note also that if ℓ is even, $\alpha A^{\ell/2} = -\alpha$ for *all* non-zero $\alpha \in \mathbb{Z}_p^r$. On the other hand, if $p = 2$, every non-zero vector (being its own inverse under the operation of addition) satisfies the requirement. In either (even or odd p) case, since A is irreducible, it does not admit invariant proper subspaces of $\mathbb{Z}_p^r$, and therefore the first r elements $\alpha, \alpha A, \ldots, \alpha A^{r-1}$ must be independent as vectors in $\mathbb{Z}_p^r$ and thus generate $\mathbb{Z}_p^r$.

It remains to prove the claims about the types of the obtained maps. Recall that G, being the orientation preserving automorphism group of $\mathcal{M}$, is isomorphic to $\langle \rho, \lambda \mid \rho^\ell, \lambda^2, (\lambda\rho)^m, \ldots \rangle$, and thus, in order to determine the face lengths m of our maps, we should determine the involutions of G. This turns out to be an easy exercise, the proof of which is left to the reader:

1. If $p \geq 3$, the set of involutions of G consists of the elements $\{(-\mathbb{I}_r, \beta) \mid \beta \in \mathbb{Z}_p^r\}$;
2. if $p = 2$, the set of involutions of G consists of the elements $\{(\mathbb{I}_r, \beta) \mid \beta \in \mathbb{Z}_2^r\}$.

Thus, $G = \langle (-\mathbb{I}_r, \beta), (A, \vec{\mathbf{0}}) \mid \beta \in \mathbb{Z}_p^r,\ \beta \neq \vec{\mathbf{0}} \rangle$ for $p \geq 3$, and $G = \langle (\mathbb{I}_r, \beta), (A, \vec{\mathbf{0}}) \mid \beta \in \mathbb{Z}_2^r,\ \beta \neq \vec{\mathbf{0}} \rangle$ when $p = 2$.

Since ℓ is obviously the true order of A, the statement can be proved by determining the order of the products $(-\mathbb{I}_r, \beta) \cdot (A, \vec{\mathbf{0}})$ or $(\mathbb{I}_r, \beta) \cdot (A, \vec{\mathbf{0}})$ (which fortunately does not depend on the choice of β).

First, assume $p \geq 3$. In this case, $(-\mathbb{I}_r, \beta) \cdot (A, \vec{\mathbf{0}}) = (-A, \beta A)$. If we denote the order of $(-A, \beta A)$ by m, then (depending on the parity of m)

$$(-A, \beta A)^m = (-A^m, \beta(A^m - A^{m-1} + \cdots - A^2 + A)) = (\mathbb{I}_r, \vec{\mathbf{0}}), \quad m \text{ odd},$$

$$(-A, \beta A)^m = (A^m, -\beta(A^m - A^{m-1} + \cdots + A^2 - A)) = (\mathbb{I}_r, \vec{\mathbf{0}}), \quad m \text{ even}.$$

Thus, if m is odd, $-A^m = \mathbb{I}_r$ and $\beta(A^m - A^{m-1} + \cdots - A^2 + A) = \vec{\mathbf{0}}$. Since -1 is not an eigenvalue of A, the matrix $A + \mathbb{I}_r$ is invertible. Hence,

$$\vec{\mathbf{0}} = A^m + I = (A + I)(A^{m-1} - A^{m-2} + \cdots - A + \mathbb{I}_r),$$

and therefore $A^{m-1} - A^{m-2} + \cdots - A + \mathbb{I}_r = \vec{\mathbf{0}}$ and $\alpha(A^m - A^{m-1} + \cdots - A^2 + A) = \vec{\mathbf{0}}$, for any positive integer m satisfying the condition $-A^m = \mathbb{I}_r$. The condition $-A^m = \mathbb{I}_r$ yields $m = \frac{\ell}{2}$, and so $\ell \equiv 2 \bmod 4$, and the type of the map is $(\ell, \frac{\ell}{2})$.

If m is even, then $A^m = \mathbb{I}_r$ and $\beta(A^m - A^{m-1} + \cdots + A^2 - A) = \vec{\mathbf{0}}$. In this case, $\vec{\mathbf{0}} = A^m - \mathbb{I}_r = (A - I)(A^{m-1} - A^{m-2} + \cdots + A - \mathbb{I}_r)$ and so $A^{m-1} - A^{m-2} + \cdots + A - \mathbb{I}_r = \vec{\mathbf{0}}$ (since $A - \mathbb{I}_r$ is invertible), which implies $\alpha(A^m - A^{m-1} + \cdots + A^2 - A) = \vec{\mathbf{0}}$ for any positive integer m satisfying the condition $A^m = \mathbb{I}_r$. The minimal solution of the identity $A^m = \mathbb{I}_r$ is $m = \ell$, and we conclude that the type of the map is (ℓ, ℓ).

Considering the case $p = 2$, the product of $(\mathbb{I}_r, \beta)$ and $(A, \vec{\mathbf{0}})$ is $(A, \beta A)$. Assume

$$(A, \beta A)^m = (A^m, \beta(A^m + A^{m-1} + \cdots + A^2 + A)) = (\mathbb{I}_r, \vec{\mathbf{0}}).$$

Because $A^\ell = \mathbb{I}_r$, one can get $\beta(A^{\ell-1} + \cdots + A + \mathbb{I}_r) = \vec{\mathbf{0}}$. So, in this case $m = \ell$, and the type of the map is (ℓ, ℓ). □

While every choice of an irreducible matrix $A \in \mathrm{GL}(r, p)$ together with any choice of a non-zero $\alpha \in \mathbb{Z}_p^r$ having the property that $\{\alpha, \alpha A, \ldots, \alpha A^{\ell-1}\}$ is closed under taking inverses gives rise to an orientably-regular map, and all the maps we are after in this section come in this way, in order to obtain a classification, we need to determine the isomorphisms classes of these maps. Fortunately, when dealing with normal Cayley maps, we can rely on the following theorem.

Theorem 2.3 ([27], Theorem 6.2) *Let $\mathcal{M}_1 = CM(H_1, \pi_1)$ and $\mathcal{M}_2 = CM(H_2, \pi_2)$ be balanced Cayley maps of valence d. Then $\mathcal{M}_1$ is isomorphic to $\mathcal{M}_2$ if and only if there exists a group isomorphism $\varphi : H_1 \to H_2$ such that $\varphi(x_i^{(1)}) = x_{i+t}^{(2)}$, for all $1 \le i \le d$, and some $1 \le t \le d$.*

Thus, two Cayley maps $\mathcal{M}_1, \mathcal{M}_2$, of the type described in Theorem 2.2, based on irreducible matrices A_1, A_2 of order ℓ, and non-zero vectors α_1, α_2, are isomorphic if and only if there exists a $B \in \mathrm{GL}(r, p)$ such that $\alpha_1 A_1^i B = \alpha_2 A_2^{i+t}$, for all $0 \le i \le \ell - 1$, and some fixed t. Recall that the r vectors $\alpha_1, \alpha_1 A_1, \ldots, \alpha_1 A_1^{r-1}$ as well as any r consecutive vectors $\alpha_2 A_2^t, \alpha_2 A_2^{t+1}, \ldots, \alpha_2 A_2^{t+r-1}$ form bases for $\mathbb{Z}_p^r$, and hence the equations $\alpha_1 A_1^i B = \alpha_2 A_2^{i+t}$, $0 \le i \le r$, uniquely determine the matrix B. If we let $m_{A_i}(x) = a_{i,0} + a_{i,1}x + \ldots + a_{i,r-1}x^{r-1} + x^r$ denote the monic minimal polynomials for A_i, $i \in \{1, 2\}$, we obtain $\alpha_i A_i^r = \alpha_i(-a_{i,0}\mathbb{I}_r - a_{i,1}A - \ldots - a_{i,r-1}A^{r-1})$. In order for B to define an automorphism between $\mathcal{M}_1$ and $\mathcal{M}_2$, the identity $\alpha_1 A_1^r B = \alpha_2 A_2^{r+t}$ together with the fact that both sets $\alpha_1, \alpha_1 A_1, \ldots, \alpha_1 A_1^{r-1}$ and $\alpha_2 A_2^t, \alpha_2 A_2^{t+1}, \ldots, \alpha_2 A_2^{t+r-1}$ form bases, implies the equality between the two minimal polynomials. This yields the following.

Lemma 2.4 *Let $A_1, A_2 \in \mathrm{GL}(p, r)$ be two irreducible matrices of order ℓ, and $\mathcal{M}_1, \mathcal{M}_2$ be the corresponding maps as in Theorem 2.2. Then the following three statements are equivalent,*

(1) $\mathcal{M}_1 \cong \mathcal{M}_2$;

(2) $m_{A_1}(x) = m_{A_2}(x)$;

(3) $A_2 = A_1^{p^s}$ *for some* $1 \le s \le r$.

Proof. From the result of linear algebra, two irreducible matrices $A_1, A_2 \in \mathrm{GL}(p, r)$ are conjugate if and only if they have the same monic minimal polynomial. $\square$

Theorem 2.5 *Let p be a prime, and $r \geq 1$. If $p = 2$, the isomorphism classes of orientably-regular maps $\mathcal{M}$ having p^r vertices and satisfying the property that the action of $\mathrm{Aut}^+\mathcal{M}$ on the vertices of $\mathcal{M}$ is quasiprimitive of type* **HA** *are in an one-to-one correspondence with the set of irreducible monic polynomials over $\mathbb{Z}_p$ of degree r. If $p \geq 3$, the one-to-one correspondence is with the set of irreducible monic polynomials over $\mathbb{Z}_p$ of degree r having the property that their associated matrix is of even order.*

Proof. For $p(x) \in \mathbb{Z}_p[x]$ irreducible of degree r, let A be the matrix associated with $p(x)$, and assume that the order ℓ of A is even when p is odd. Let α be any non-zero vector from $\mathbb{Z}_p^r$. Since, in either case, the orbit of α under A is closed under taking additive inverses, the matrix A and the vector α give rise to an orientably-regular map $CM(\mathbb{Z}_p^r, (\alpha, \alpha A, \ldots, \alpha A^{\ell-1}))$ that has the desired properties (Theorem 2.2). For different irreducible polynomials, these maps are pairwise non-isomorphic (Lemma 2.4), while any other map $\mathcal{M}$ with the properties described in our theorem is isomorphic to exactly one of these maps - the one with the same minimal polynomial $p(x)$ (Theorem 2.2 and Lemma 2.4). $\square$

As a result of Theorem 2.5 and Lemma 2.4, one can easily get the following Corollary 2.6 and we omit the proof.

Corollary 2.6 *Let $G = \mathbb{Z}_p^r \rtimes \mathbb{Z}_\ell$ be a quasiprimitive group of type* **HA***, where $\mathbb{Z}_\ell$ is generated by an irreducible $A \in \mathrm{GL}(r, p)$, then there are $\phi(\ell)/r$ non-isomorphic normal Cayley maps having G as the automorphism group, where ϕ is the Euler's totient function.*

We close the subsection with the classification of those of the above maps which are reflexible.

Theorem 2.7 *Let $G = \mathbb{Z}_p^r \rtimes \mathbb{Z}_\ell$ be a quasiprimitive group of type* **HA** *and let $\mathbb{Z}_\ell$ be generated by an irreducible $A \in GL(r, p)$. Then, the Cayley map $\mathcal{M} = CM(\mathbb{Z}_p^r, \pi)$, where $\pi = (\alpha, \alpha A, \ldots, \alpha A^{\ell-1})$ for some non-zero vector $\alpha \in \mathbb{Z}_p^r$ is reflexible if and only if A and A^{-1} have the same minimal polynomial. In other words, $\mathcal{M}$ is reflexible if and only if r is even and $\ell | p^{\frac{r}{2}} + 1$.*

Proof. By definition, the orientably-regular $\mathcal{M} = CM(\mathbb{Z}_p^r, (\alpha, \alpha A, \ldots, \alpha A^{\ell-1}))$ is reflexible if and only if it is isomorphic to the orientably-regular $CM(\mathbb{Z}_p^r, (\alpha, \alpha A^{-1}, \ldots, \alpha (A^{-1})^{\ell-1}))$. According to Lemma 2.4, $\mathcal{M}$ is reflexible if and only if the minimal polynomial of A is the same as the minimal polynomial of A^{-1}, and if and only if $A^{-1} = A^{p^s}$ for some $1 \leq s \leq r$. The equality $A^{p^s} = A^{-1}$ implies $\ell | p^s + 1$. Because ℓ is a primitive divisor of $p^r - 1$, it follows that $r = 2s$. $\square$

2.2 Regular maps of type HA

Let us now consider the case in which the entire automorphism group of a map acts quasiprimitively on the vertices of the map in an action of type **HA**.

Thus, let G be the automorphism group of the regular map $\mathcal{M} = \mathrm{Map}(G; x, y, z)$ represented in its action on the vertices of $\mathcal{M}$, and assume the action of G to be quasiprimitive of type **HA**. The vertices of $\mathcal{M}$ can then be identified with $\mathbb{Z}_p^r$, and $G = \mathbb{Z}_p^r \rtimes H$, where H is an irreducible subgroup of $\mathrm{GL}(r,p)$. Since H is also the stabilizer of $\vec{\mathbf{0}}$, and $\mathcal{M}$ is regular, $H \cong \mathbb{D}_\ell$, the dihedral group of order 2ℓ. Hence, $G = \mathbb{Z}_p^r \rtimes \langle A, B\rangle$, $A, B \in \mathrm{GL}(r,p)$, $A^\ell = B^2 = \mathbb{I}_r$, $BAB = A^{-1}$, with $\langle A, B\rangle$ irreducible, i.e., having no non-trivial invariant subspaces.

On the other hand, $G = \langle x, y, z\rangle$, with both y and z fixing $\vec{\mathbf{0}}$, therefore $H = \langle y, z\rangle = \langle AB, B\rangle$. The element x must be an involution that does not belong to H and commutes with AB. If we return to the notation of elements in G from the beginning of Subsection 2.1, $x = (C, \alpha)$, $G = \langle (C,\alpha), (AB, \vec{\mathbf{0}}), (B, \vec{\mathbf{0}})\rangle$, and

$$(C,\alpha)^2 = (AB,\vec{\mathbf{0}})^2 = (B,\vec{\mathbf{0}})^2 = ((C,\alpha)(AB,\vec{\mathbf{0}}))^2 = ((AB,\vec{\mathbf{0}})(B,\vec{\mathbf{0}}))^\ell = ((C,\alpha)(B,\vec{\mathbf{0}}))^m = 1$$

for integers $\ell, m \geq 3$, representing the degree and face-length of $\mathcal{M}$, respectively. Consequently,

$$(AB)^2 = B^2 = C^2 = \mathbb{I}_r, CAB = ABC, \alpha C = -\alpha, \alpha AB = \alpha,$$

but $BC \neq CB$. Since (C,α) is an involution, either $C = -\mathbb{I}_r$ or $C = A^i B$ for some $1 \leq i \leq \ell - 1$. If $C = A^i B$, then the identity $CAB = ABC$ yields $i = 1$ when $p = 2$ or $i = \frac{\ell}{2} + 1$ and $C = -AB$ when $p \geq 3$ (since ℓ is necessarily even in this case). We have proved the following.

Lemma 2.8 *The set of involutions in G that commute with $(AB, \vec{\mathbf{0}})$ consists of the elements*

$$\{(\mathbb{I}_r, \alpha), (AB, \alpha) \mid \alpha AB = \alpha\},$$

when $p = 2$ and

$$\{(-\mathbb{I}_r, \alpha), (-AB, \alpha) \mid \alpha AB = \alpha\},$$

when $p \geq 3$.

There are two ways in which $\langle A, B\rangle$ can be irreducible in its action on $\mathbb{Z}_p^r$. Either $\langle A\rangle$ is already irreducible, or A admits a non-trivial invariant subspace $\mathbb{V}$ which is mapped to its complementary space by B. To reflect the above two possibilities, we divide the rest of this subsection into two cases, depending on whether $\langle A\rangle$ is irreducible or not.

Case 1. Assume $\langle A\rangle$ acts irreducibly on $\mathbb{Z}_p^r$.

Theorem 2.9 *Let G be a quasiprimitive group of type* **HA**. *If*

$$G = \mathbb{Z}_p^r \rtimes \mathbb{D}_\ell = \langle (C,\alpha), (AB, \vec{\mathbf{0}}), (B, \vec{\mathbf{0}})\rangle$$

as discussed at the beginning of this section, and $\langle A\rangle$ is irreducible, then

(1) *when $p = 2$, G is the automorphism group of a nonorientably-regular map of type $(\ell, 4)$ if $(C, \alpha) = (\mathbb{I}_r, \alpha)$, and is the automorphism group of an orientable and reflexible regular map of type (ℓ, ℓ) if $(C, \alpha) = (AB, \alpha)$, where $\alpha \neq \vec{\mathbf{0}}$ and $\alpha AB = \alpha$;*

(2) *when $p \geq 3$, G is the automorphism group of a nonorientably-regular map of type $(\ell, 2p)$ when $(C, \alpha) = (-\mathbb{I}_r, \alpha)$, and is the automorphism group of an orientable and reflexible regular map of type $(\ell, \frac{\ell}{2})$ if $\ell \equiv 2 \bmod 4$ and of type (ℓ, ℓ) if $\ell \equiv 0 \bmod 4$ when $(C, \alpha) = (-AB, \alpha)$, where $\alpha \neq \vec{\mathbf{0}}$ and $\alpha AB = \alpha$.*

Proof. We only prove the result for $p \geq 3$ and one may prove quite similarly for $p = 2$. If $(C, \alpha) = (-\mathbb{I}_r, \alpha)$ for some $\alpha \neq 0$ and $\alpha AB = \alpha$, then, $(B, \vec{\mathbf{0}})(-\mathbb{I}_r, \alpha) = (-B, \alpha)$. A direct calculation shows that $(-B, \alpha)^2 = (\mathbb{I}_r, -\alpha B + \alpha), (-B, \alpha)^3 = (-B, -\alpha B + 2\alpha), (-B, \alpha)^4 = (\mathbb{I}_r, -2\alpha B + 2\alpha), (-B, \alpha)^5 = (-B, -2\alpha B + 3\alpha), (-B, \alpha)^6 = (\mathbb{I}_r, -3\alpha B + 3\alpha), \cdots$. Because the eigenvalues of B are ± 1 and B doesn't have common eigenvectors with AB, it follows that the order of $(-B, \alpha)$ is $2p$. So, the type of the regular map is $(\ell, 2p)$.

Set $x = (-I, \alpha), y = (AB, \vec{\mathbf{0}}), z = (B, \vec{\mathbf{0}})$, then $xy = (-AB, \alpha), yz = (A, \vec{\mathbf{0}}), xz = (-B, \alpha B)$. $zyxyyz = (-BA, \alpha A), zx(-BA, \alpha A)xz = (-AB, \alpha A^{-1} - \alpha + \alpha A), (-AB, \alpha A^{-1} - \alpha + \alpha A)(-AB, \alpha) = (I, -\alpha A + 2\alpha - \alpha A^{-1})$. Suppose $-\alpha A + 2\alpha - \alpha A^{-1} = \vec{\mathbf{0}}$, then $|I - A| = 0$ which contradicts to the irreducibility assumption of A. As a result, the subgroup H generated by $(I, -\alpha A + 2\alpha - \alpha A^{-1})$ and $(A, \vec{\mathbf{0}})$ is of index two in G because of the irreducibility of A. However, it is clear that xz is not included in H. So, $G = \langle xy, yz \rangle$ which implies the nonorientability of the map.

If $(C, \alpha) = (-AB, \alpha)$ for some $\alpha \neq 0$ and $\alpha AB = \alpha$, then $(B, \vec{\mathbf{0}})(-AB, \alpha) = (-A^{-1}, \alpha)$. Note that in this case, ℓ is even. By a similar enumeration,

$$(-A^{-1}, \alpha)^m = ((-A^{-1})^m, \alpha((-A^{-1})^{m-1} + (-A^{-1})^{m-2} + \cdots + (-A^{-1}) + \mathbb{I}_r)).$$

Assume the order of $(-A^{-1}, \alpha)$ is m, then one can get $(-A^{-1})^m = \mathbb{I}_r$ and

$$\alpha((-A^{-1})^{m-1} + (-A^{-1})^{m-2} + \cdots + (-A^{-1}) + \mathbb{I}_r) = \vec{\mathbf{0}}.$$

When $\ell \equiv 2 \bmod 4$, the smallest integer satisfying $(-A^{-1})^m = \mathbb{I}_r$ is $m = \frac{\ell}{2}$. In this case, the equality $\mathbf{0} = (A^{-1})^{\frac{\ell}{2}} + \mathbb{I}_r = (A^{-1} + I)((A^{-1})^{\frac{\ell}{2}-1} - (A^{-1})^{\frac{\ell}{2}-2} + \cdots - A^{-1} + \mathbb{I}_r)$ and the fact that the matrix $A^{-1} + \mathbb{I}_r$ is invertible imply that $(A^{-1})^{\frac{\ell}{2}-1} - (A^{-1})^{\frac{\ell}{2}-2} + \cdots - A^{-1} + \mathbb{I}_r = \mathbf{0}$. So, in this case, the type of the map is $(\ell, \frac{\ell}{2})$. Otherwise, the smallest integer satisfying $(-A^{-1})^m = \mathbb{I}_r$ is $m = \ell$. In this case, from the relation $\mathbf{0} = (A^{-1})^{\ell} - \mathbb{I}_r = (A^{-1} + \mathbb{I}_r)((A^{-1})^{\ell-1} - (A^{-1})^{\ell-2} + \cdots + A^{-1} - \mathbb{I}_r)$ and the fact that the matrix $A^{-1} + \mathbb{I}_r$ is invertible, follows that $(A^{-1})^{\ell-1} - (A^{-1})^{\ell-2} + \cdots + A^{-1} - \mathbb{I}_r = \mathbf{0}$. So, in this situation, the type of the map is (ℓ, ℓ).

Set $x = (-AB, \alpha), y = (AB, \vec{\mathbf{0}}), z = (B, \vec{\mathbf{0}})$, then $xy = (-I, \alpha), yz = (A, \vec{\mathbf{0}})$. It is clear that the subgroup generated by xy and yz is of index two in G. So, in this case G is the automorphism group of an orientable and reflexible regular map. □

As for the number of non-isomorphic regular maps in this case, one may look back to Section 1 for details and we only state the result in Theorem 2.10.

Theorem 2.10 *Let G be a quasiprimitive group of type* **HA**. *If*

$$G = \mathbb{Z}_p^r \rtimes \mathbb{D}_\ell = \langle (C, \alpha), (AB, \vec{\mathbf{0}}), (B, \vec{\mathbf{0}}) \rangle$$

as discussed at the beginning of this section, and $\langle A \rangle$ is irreducible, then there are $2\phi(\ell)/r$ non-isomorphic regular maps having G as the automorphism group, half of them are non-orientable and half of them are orientable and reflexible, where ϕ is the Euler's totient function.

Case 2. Assume $\langle A \rangle$ is reducible in its action on $\mathbb{Z}_p^r$.

We claim that $p \neq 2$. If p were equal to 2, the set $\mathbb{Z}_p^r - \vec{\mathbf{0}}$ would be of odd cardinality, and hence the action of $\mathbb{D}_\ell$ on $\mathbb{Z}_p^r - \vec{\mathbf{0}}$ would have at least one fixed point $\delta \neq \vec{\mathbf{0}}$. The 1-dimensional subspace generated by δ would then be an invariant subspace of $\mathbb{D}_\ell$, a contradiction. So, in the remaining part of this section, *we shall assume* $p \geq 3$.

If A is reducible, it has two complementary invariant subspaces of equal dimensions, $\mathbb{V}_1$ and $\mathbb{V}_2$. In this case, r is even, and $\mathbb{Z}_p^r = \mathbb{V}_1 \oplus \mathbb{V}_2$, and $\dim \mathbb{V}_1 = \dim \mathbb{V}_2 = \frac{r}{2}$. Without loss of generality, we may assume that $A = diag(P_1, P_2)$, where P_1 and P_2 are $\frac{r}{2} \times \frac{r}{2}$ invertible matrices. If we denote the minimal polynomials of P_1 and P_2 by $m_1(x)$ and $m_2(x)$, respectively, then both of them are irreducible over $\mathbb{Z}_p$ of degree $\frac{r}{2}$. Let $m(x)$ be the minimal polynomial of A. The following Lemma 2.11 will give the relation between $m(x)$ and $m_1(x), m_2(x)$.

Lemma 2.11 *Under the assumption and notation of the preceding paragraph, the set $\{\alpha, \alpha A, \alpha A^2, \cdots, \alpha A^{r-1}\}$ must form a basis of $\mathbb{Z}_p^r$ where α is an eigenvector of AB associated with the eigenvalue* 1*, and $m(x) = m_1(x)m_2(x)$.*

Proof. Since the action of $\mathbb{D}_\ell$ on $\mathbb{Z}_p^r$ does not admit non-trivial invariant subspaces, given any non-zero $\alpha \in \mathbb{Z}_p^r$, the set $\{\alpha A^i B^j \mid 0 \leq i \leq \ell - 1, 0 \leq j \leq 1\}$ contains a basis of $\mathbb{Z}_p^r$. In particular, if α is an eigenvector of AB associated with the eigenvalue 1 (recall that $(AB)^2 = \mathbb{I}_r$), it is easy to see that $\{\alpha A^i B^j \mid 0 \leq i \leq \ell - 1, 0 \leq j \leq 1\} = \{\alpha A^i \mid 0 \leq i \leq \ell - 1\}$, which implies that the set $\{\alpha, \alpha A, \alpha A^2, \cdots, \alpha A^{r-1}\}$ must form a basis of $\mathbb{Z}_p^r$. It follows that the degree of $m(x)$ must be r, $m_1(x)$ and $m_2(x)$ must be coprime, and $m(x) = m_1(x)m_2(x)$. □

Since, $\mathbb{V}_1 A = \mathbb{V}_1$ and $\mathbb{V}_2 A = \mathbb{V}_2$, it follows that $\mathbb{V}_1 AB = \mathbb{V}_1 B$ and $\mathbb{V}_2 AB = \mathbb{V}_2 B$. Suppose that $\mathbb{V}_1 B = \mathbb{W}_1 \oplus \mathbb{W}_2$, where $\mathbb{W}_1 \subseteq \mathbb{V}_1$ and $\mathbb{W}_2 \subseteq \mathbb{V}_2$. Then $\mathbb{W}_1 \oplus \mathbb{W}_2 = \mathbb{V}_1 B = \mathbb{V}_1 AB = \mathbb{V}_1 BA^{-1} = \mathbb{W}_1 A^{-1} \oplus \mathbb{W}_2 A^{-1}$. Thus, $\mathbb{W}_1 = \mathbb{W}_1 A^{-1}$ and $\mathbb{W}_2 = \mathbb{W}_2 A^{-1}$. However, $\mathbb{V}_1$ and $\mathbb{V}_2$ are the only non-trivial invariant subspaces for A and A^{-1}, therefore $\mathbb{W}_1 = \langle \vec{\mathbf{0}} \rangle$ or $\mathbb{W}_1 = \mathbb{V}_1$, and $\mathbb{W}_2 = \langle \vec{\mathbf{0}} \rangle$ or $\mathbb{W}_2 = \mathbb{V}_2$. Since the dimension of $\mathbb{V}_1 B$ is $\frac{r}{2}$, exactly one of $\mathbb{W}_1, \mathbb{W}_2$ must be equal to $\langle \vec{\mathbf{0}} \rangle$. Note that $\mathbb{W}_1$ cannot be equal to $\mathbb{V}_1$, for otherwise $\mathbb{V}_1$ would be invariant with respect to both A and B. Thus, $\mathbb{W}_1 = \langle \vec{\mathbf{0}} \rangle$ and $\mathbb{W}_2 = \mathbb{V}_2$. Applying the same ideas to $\mathbb{V}_2 B$ we see that $\mathbb{V}_1 B = \mathbb{V}_1 AB = \mathbb{V}_2$ and $\mathbb{V}_2 B = \mathbb{V}_2 AB = \mathbb{V}_1$. Therefore, we can assume without loss of generality, that

$$AB = \begin{pmatrix} 0 & B_1 \\ B_2 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & C_1 \\ C_2 & 0 \end{pmatrix},$$

with the submatrices satisfying the identities $B_1B_2 = \mathbb{I}_{\frac{r}{2}}$, $C_1C_2 = \mathbb{I}_{\frac{r}{2}}$, $P_1 = B_1C_2$ and $P_2 = B_2C_1$ (recall that $A = diag(P_1, P_2)$).

Since AB and B are involutions, their two eigenvalues are 1 and -1. Their corresponding eigenvectors are described in the following lemma.

Lemma 2.12 *The vector α is an eigenvector of AB (resp. B) associated with the eigenvalue 1 if and only if $\alpha = \alpha_1 + \alpha_1 AB$ (resp. $\alpha = \alpha_1 + \alpha_1 B$) for some non-zero vector $\alpha_1 \in \mathbb{V}_1$. The vector β is an eigenvector of AB (B) associated with the eigenvalue -1 if and only if $\beta = \beta_1 - \beta_1 AB$ ($\beta = \beta_1 - \beta_1 B$) for some non-zero vector $\beta_1 \in \mathbb{V}_1$.*

Proof. Assume that $\alpha = \alpha_1 + \alpha_2$ is an eigenvector of AB belonging to the eigenvalue 1, where $\alpha_1 \in \mathbb{V}_1$ and $\alpha_2 \in \mathbb{V}_2$. Then from $\alpha AB = \alpha$, one can get $\alpha_1 AB + \alpha_2 AB = \alpha_1 + \alpha_2$. Since $\mathbb{V}_1 AB = \mathbb{V}_2$ and $\mathbb{V}_2 AB = \mathbb{V}_1$, we obtain $\alpha_2 = \alpha_1 AB$. So, $\alpha = \alpha_1 + \alpha_1 AB$. As for the other direction, it is easy to check that $\alpha = \alpha_1 + \alpha_1 AB$, for each $\vec{\mathbf{0}} \neq \alpha_1 \in \mathbb{V}_1$, is an eigenvector of A$B$ belonging to the eigenvalue 1. The other results of the lemma can be proved similarly, and we omit the details. $\square$

The condition $A^\ell = \mathbb{I}_r$ yields that $P_1^\ell = P_2^\ell = \mathbb{I}_{\frac{r}{2}}$, thus, ℓ is a primitive divisor of $p^{\frac{r}{2}} - 1$. Therefore, $P_1^{\frac{\ell}{2}} = \pm\mathbb{I}_{\frac{r}{2}}$ and $P_2^{\frac{\ell}{2}} = \pm\mathbb{I}_{\frac{r}{2}}$ (with at most one of them equal to $\mathbb{I}_{\frac{r}{2}}$). However, the equality $A^{\frac{\ell}{2}}B = BA^{\frac{\ell}{2}}$ shows that $P_1^{\frac{\ell}{2}} = P_2^{\frac{\ell}{2}} = -\mathbb{I}_{\frac{r}{2}}$, and consequently, $A^{\frac{\ell}{2}} = -\mathbb{I}_{\frac{r}{2}}$. The proof of the following Theorem 2.13 is analogous to the proof of Theorem 2.9.

Theorem 2.13 *Let G be a quasiprimitive group of type* **HA**. *If*

$$G = \mathbb{Z}_p^r \rtimes \mathbb{D}_\ell = \langle (C, \alpha), (AB, \vec{\mathbf{0}}), (B, \vec{\mathbf{0}}) \rangle$$

and $\langle A \rangle$ is reducible, then G is the automorphism group of a nonorientably-regular map of type $(\ell, 2p)$ when $(C, \alpha) = (-\mathbb{I}_r, \alpha)$ for some $\alpha \neq \vec{\mathbf{0}}$ and $\alpha AB = \alpha$; and if $(C, \alpha) = (-AB, \alpha)$ for some $\alpha \neq \vec{\mathbf{0}}$ and $\alpha AB = \alpha$, then G is the automorphism group of an orientable and reflexible regular map of type $(\ell, \frac{\ell}{2})$ when $\ell \equiv 2 \bmod 4$, and of type (ℓ, ℓ) when $\ell \equiv 0 \bmod 4$.

Proof. The calculation is almost the same as in the proof of Theorem 2.9. The only difference is in showing the non-orientability. To show the nonorientability of the map when $x = (-I, \alpha), y = (AB, \vec{\mathbf{0}}), z = (B, \vec{\mathbf{0}})$. The element $(I, -\alpha A + 2\alpha - \alpha A^{-1})$ belongs to $H = \langle xy, yz \rangle$. As a result, the subgroup H generated by $(I, -\alpha A + 2\alpha - \alpha A^{-1})$ and $(A, \vec{\mathbf{0}})$ is still of index two in G although A is reducible. In fact, $-\alpha A + 2\alpha - \alpha A^{-1}$ is an eigenvector of AB belonging to the eigenvalue 1. So, according to the discussion preceding to the theorem, $(I, -\alpha A + 2\alpha - \alpha A^{-1})$ and $(A, \vec{\mathbf{0}})$ can generate $\mathbb{Z}_p^r \rtimes \mathbb{Z}_\ell$. $\square$

Because $\mathbb{Z}_p^r$ is a normal Hall subgroup of G, employing Hall's theorem yields that any two dihedral subgroups of G are conjugate. Since we assume that the dihedral subgroup $\langle AB, B \rangle$ is maximal, the two generating involutions of the dihedral subgroup together with any involution outside the dihedral subgroup which commutes with one of the two involutions generate the entire group G. Taking an element $(D,\xi)\in G$, the conjugates of $(AB,0)$ and $(B,0)$ under (D,ξ) yield involutions generating a dihedral subgroup. The two conjugates are the elements $(D^{-1}ABD, -\xi D^{-1}ABD+\xi), (D^{-1}BD, -\xi D^{-1}BD+\xi)$, and an easy calculation shows that any involution not belonging to $\langle (D^{-1}ABD, -\xi D^{-1}ABD+\xi), (D^{-1}BD, -\xi D^{-1}BD+\xi)\rangle$ but commuting with $(D^{-1}ABD, -\xi D^{-1}ABD+\xi)$ is a conjugate of $(-\mathbb{I}_r,\alpha)$ or of $(-AB,\alpha)$ under (D,ξ); for some non-zero α with $\alpha AB=\alpha$. These arguments together with Theorem 2.13 show that in order to classify (up to isomorphism) regular maps whose automorphism group is G, we only need to calculate the orbits of $\mathrm{Aut}(G)$ in its action on the generating sets $\{(-\mathbb{I}_r,\alpha),(AB,\vec{\mathbf{0}}),(B,\vec{\mathbf{0}})\}$ or $\{(-AB,\alpha),(AB,\vec{\mathbf{0}}),(B,\vec{\mathbf{0}})\}$.

Lemma 2.14 *Given any two generating sets $\{(-\mathbb{I}_r,\alpha),(AB,\vec{\mathbf{0}}),(B,\vec{\mathbf{0}})\}$ and $\{(-\mathbb{I}_r,\beta),(AB,\vec{\mathbf{0}}),(B,\vec{\mathbf{0}})\}$, where $\alpha AB=\alpha$ and $\beta AB=\beta$, there is an automorphism of G that maps $(-\mathbb{I}_r,\alpha)$ to $(-\mathbb{I}_r,\beta)$ and fixes both $(AB,\vec{\mathbf{0}})$ and $(B,\vec{\mathbf{0}})$.*

Proof. Recall that $\mathbb{Z}_p^r=V_1\oplus V_2$. Thus, given any basis $\alpha_1,\alpha_2,\cdots,\alpha_{\frac{r}{2}}$ of V_1, the set

$$\alpha_1,\alpha_2,\cdots,\alpha_{\frac{r}{2}},\alpha_1B,\alpha_2B,\cdots,\alpha_{\frac{r}{2}}B$$

forms a basis for $\mathbb{Z}_p^r$. Now, with respect to this basis, we can assume that

$$B=\begin{pmatrix}0 & I\\ I & 0\end{pmatrix},\quad AB=\begin{pmatrix}0 & C_1\\ C_1^{-1} & 0\end{pmatrix},\ \text{and consequently}\ A=\begin{pmatrix}C_1 & 0\\ 0 & C_1^{-1}\end{pmatrix}.$$

Let $\alpha=\delta+\delta B$ be an eigenvector of B associated with the eigenvalue 1, where $\delta\in V_1$. We can assume that $\delta=(\vec{X},\vec{\mathbf{0}})$, where $\vec{X}$ is a vector of dimension $\frac{r}{2}$, and as a result, $\alpha=(\vec{X},\vec{X})$. Recall that C_1 and C_1^{-1} act irreducibly on V_1 and V_2, respectively. Hence, $\{(\vec{X},\vec{\mathbf{0}}),(\vec{X}C_1,\vec{\mathbf{0}}),\cdots,(\vec{X}C_1^{\frac{r}{2}-1},\vec{\mathbf{0}})\}$ and $\{(\vec{\mathbf{0}},\vec{X}),(\vec{\mathbf{0}},\vec{X}C_1^{-1}),\cdots,(\vec{\mathbf{0}},\vec{X}C_1^{-\frac{r}{2}+1})\}$ are bases of V_1 and V_2, respectively. Let $\beta=(\vec{Y},\vec{Y})$ be another eigenvector of B associated with the eigenvalue 1. Then, there exist λ_i,μ_i, $0\le i\le\frac{r}{2}-1$, such that

$$\begin{array}{rcll}\vec{Y} & = & \lambda_0\vec{X}+\lambda_1\vec{X}C_1+\cdots+\lambda_{\frac{r}{2}-1}\vec{X}C_1^{\frac{r}{2}-1} & =\vec{X}f_1(C_1)\\ & = & \mu_0\vec{X}+\mu_1\vec{X}C_1^{-1}+\cdots+\mu_{\frac{r}{2}-1}\vec{X}C_1^{-\frac{r}{2}+1} & =\vec{X}f_2(C_1^{-1})\end{array},$$

where $f_1(x)=\lambda_0+\lambda_1x+\cdots+\lambda_{\frac{r}{2}-1}x^{\frac{r}{2}-1}$ and $f_2(x)=\mu_0+\mu_1x+\cdots+\mu_{\frac{r}{2}-1}x^{\frac{r}{2}-1}$. Since $f_1(C_1)$ commutes with C_1, using Lemma 2.12 from [1], we have $f_1(C_1)=g(C_1^{-1})$ for some polynomial $g(x)\in F_p(x)$. This implies the identity $\vec{X}f_1(C_1)=\vec{X}g(C_1^{-1})=\vec{X}f_2(C_1^{-1})$. However, using Schur's Lemma asserting that the endomorphism ring of an irreducible module is a division algebra, we obtain that $g(C_1^{-1})=f_2(C_1^{-1})$, which implies the equality $f_1(C_1)=f_2(C_1^{-1})$.

Finally, let $P=\begin{pmatrix}f_1(C_1^{-1}) & 0\\ 0 & f_2(C_1)\end{pmatrix}$. It is easy to check that P commutes with both AB and B and $\alpha P=\beta$. If we define $\sigma:(D,\xi)\mapsto(D,\xi P)$, it is easy to check that σ is an automorphism of G which maps $(-\mathbb{I}_r,\alpha)$ to $(-\mathbb{I}_r,\beta)$ and fixes both $(AB,\vec{\mathbf{0}})$ and $(B,\vec{\mathbf{0}})$. □

Corollary 2.15 *The automorphism σ given in the proof of Theorem 2.14 also maps $(-AB, \alpha)$ to $(-AB, \beta)$ and fixes both $(AB, \vec{\mathbf{0}})$ and $(B, \vec{\mathbf{0}})$.*

Given any integer i which is a power of p and is co-prime with ℓ, the matrices A and A^i are conjugate, and thus $A^i = Q^{-1}AQ$ for some Q. Note that this is in fact the Galois automorphism and $B = Q^{-1}BQ$. One can very easily get the following Lemma 2.16 and we omit the proof.

Lemma 2.16 *If Q is a matrix that satisfies $A^i = Q^{-1}AQ$, then the mapping $\tau : (D, \xi) \mapsto (Q^{-1}DQ, \xi Q)$ is an automorphism of G.*

Theorem 2.17 *Let G be a quasiprimitive group of type* **HA** *and $G = \mathbb{Z}_p^r \rtimes \langle A, B\rangle$, $A, B \in \mathrm{GL}(r, p)$, $A^\ell = B^2 = \mathbb{I}_r$, $BAB = A^{-1}$, with $\langle A, B\rangle$ irreducible. Then, there are $\frac{4\phi(\ell)}{r}$ non-isomorphic regular maps with automorphism group G, half of them are non-orientable and half are orientable and reflexible, where ϕ is the Euler totient function.*

Proof. From Theorem 2.13 and the discussion preceding the theorem, we know that the number of non-isomorphic non-orientable regular maps with automorphism group G is equal to the orbits of $\mathrm{Aut}(G)$ acting on the generating subsets $\{(-\mathbb{I}_r, \alpha), (AB, \vec{\mathbf{0}}), (B, \vec{\mathbf{0}})\}$, and the number of non-isomorphic orientably-regular and reflexible maps is equal to the orbits of $\mathrm{Aut}(G)$ acting on $\{(-AB, \alpha), (AB, \vec{\mathbf{0}}), (B, \vec{\mathbf{0}})\}$. We take two generating subsets of G, say $\{(-\mathbb{I}_r, \alpha), (AB, \vec{\mathbf{0}}), (B, \vec{\mathbf{0}})\}$ and $\{(-\mathbb{I}_r, \beta), (A^iB, \vec{\mathbf{0}}), (B, \vec{\mathbf{0}})\}$ for some i coprime with ℓ, where $\alpha AB = \alpha$ and $\beta A^iB = \beta$. According to Lemma 2.16, there is an automorphism of G which maps $\{(-\mathbb{I}_r, \alpha)$ to $\{(-\mathbb{I}_r, \alpha Q)$, $(AB, \vec{\mathbf{0}})$ to $(A^iB, \vec{\mathbf{0}})$ and fixes $(B, \vec{\mathbf{0}})$ when and only if i is a power of p, where Q is an invertible matrix. And from Lemma 2.14, there is an automorphism of G which maps $\{(-\mathbb{I}_r, \alpha Q)$ to $\{(-\mathbb{I}_r, \beta)$ and fixes both $(A^iB, \vec{\mathbf{0}})$ and $(B, \vec{\mathbf{0}})$. The situation is the same when we choose the generating subset $\{(-AB, \alpha), (AB, \vec{\mathbf{0}}), (B, \vec{\mathbf{0}})\}$. While in this case G corresponds to orientable and reflexible regular maps.

Recall that the minimal polynomial of A is the product of two degree $\frac{r}{2}$ irreducible polynomials which are coprime. Thus, considering the set of integers $1 \leq i \leq \ell$ satisfying $(i, \ell) = 1$, exactly $\frac{r}{2}$ powers of A belong to the same orbit. It follows that there are $\frac{2\phi(\ell)}{r}$ non-isomorphic non-orientably regular maps, as well as $\frac{2\phi(\ell)}{r}$ non-isomorphic orientably-regular maps, where ϕ is the Euler totient function. $\square$

3 Quasiprimitive regular maps of type TW

For a description of quasiprimitive groups of twisted wreath product type (**TW**) we follow [26] and we also adopt Suzuki's notation [30] for elements and multiplication in a semidirect product.

Let P be a finite group with a subgroup Q of index k, and let T be a non-Abelian finite simple group. Further, let $\varphi : \ Q \to \mathrm{Aut}(T)$ be a group homomorphism, assigning to every $\omega \in Q$ an automorphism $t \mapsto t^{\varphi(\omega)}$ of T, such that the core of $\varphi^{-1}(\mathrm{Inn}(T))$ in P is trivial. Let $B = \{f: P \to T\}$ denote the group of all functions with domain P and with values in T, with the operation of pointwise multiplication, and with the property that $f(\alpha\omega) = f(\alpha)^{\varphi(\omega)}$ for every $\alpha \in P$ and $\omega \in Q$. This implies that functions in B are uniquely determined by their values on some Q-transversal in P, and therefore $B \cong T^k$. Finally, for every $\beta \in P$ and $f \in B$ we define a function $f^\beta: P \to T$ by letting $f^\beta(\alpha) = f(\beta\alpha)$ for every $\alpha \in P$. It can be checked that $f^\beta \in B$; moreover, for every $\beta \in P$ the mapping $\theta_\beta: B \to B,\ f \mapsto f^\beta$, well-defines an automorphism of the group B and the assignment $\theta: P \to \mathrm{Aut}(B),\ \beta \mapsto \theta_\beta$ is a group homomorphism. In what follows we will simply refer to the *action* of P on B given by $f \mapsto f^\beta$ for $\beta \in P$ and $f \in B$.

In the terminology and notation just introduced, the *twisted wreath product* $T\ \mathrm{twr}_\varphi\ P$ is defined as the semidirect product $B \rtimes_\theta P$ of B and P with respect θ. In Suzuki's notation [30], elements of the twisted wreath product will be pairs (α, f), where $\alpha \in P$ and $f \in B$, and the product of a pair of such elements is given by $(\alpha, f) \cdot (\beta, g) = (\alpha\beta, f^\beta g)$. To simplify the notation we will omit brackets and write $\alpha f \cdot \beta g = \alpha\beta f^\beta g$, with no risk of confusion; we will also leave out α or f when these stand for the identity element of P or B. Observe that in this notation we have $f^\alpha = \alpha^{-1} f \cdot \alpha$ for every $f \in B$ and $\alpha \in P$, that is, we may identify f^α with the α-conjugate of f. The twisted wreath product $T\ \mathrm{twr}_\varphi\ P = B \rtimes_\theta P$ acts on B, regarded as a set, on the *right* by sending every $f \in B$ under the action of $\beta g \in T\ \mathrm{twr}_\varphi\ P$ onto the element $f \cdot \beta g = f^\beta g \in B$. This is what makes $T\ \mathrm{twr}_\varphi\ P$ a permutation group on B of *twisted wreath product type*, or **TW** type, for short. Note that the description of the action agrees with what happens in the second coordinate of the first and the third term in the multiplication rule for $T\ \mathrm{twr}_\varphi\ P$.

Suppose now that a twisted wreath product $G = T\ \mathrm{twr}_\varphi\ P = B \rtimes P$ as above is (isomorphic to) the automorphism group of a regular map $M = \mathrm{Map}(G; x, y, z)$ of degree $k \geq 3$ such that G is quasiprimitive, of type **TW**, on the vertex set of M. This means that we may identify the vertex set of M with the set of elements of the group B and the action is as described in the previous paragraph. The group P may then be identified with the stabiliser of a vertex of M, which means that P may be assumed to be a dihedral group D_k of order $2k$ with presentation $P = \langle y, z \mid y^2, z^2, (yz)^k \rangle = \langle \lambda, \rho \mid \lambda^2, \rho^k, (\lambda\rho)^2 \rangle$ for $\lambda = y$. It follows that the third involutory generator x of G must have the form $x = \lambda\rho^j b$ or $x = \rho^j b$ for some $b \in B$ and $j \in \{0, 1, \ldots, k-1\}$; in the first case from $x^2 = 1$ we have $b^{\lambda\rho^j} b = 1$, and in the second case $x^2 = 1$ implies that either $j = 0$ and $b^2 = 1$, or $j = k/2$ for k even and $b^{\rho^{k/2}} b = 1$. Moreover, in the first case M is orientable since the subgroup $\langle yz, zx \rangle = \langle yx, yz \rangle = \langle \rho^j b, \rho \rangle = \langle \rho, b \rangle$ has index two in G, and in the second case, M is non-orientable if and only if $b \in \langle \lambda b, \rho \rangle$.

Further conditions on b are, of course, imposed by a choice of a non-trivial subgroup $Q < P$ and a homomorphism $\varphi: Q \to \mathrm{Aut}(T)$; triviality of the core of $\varphi^{-1}(\mathrm{Inn}(T))$ in P can simply be achieved by choosing *outer* automorphisms of T as non-trivial images of φ. Omitting details we note that similar conditions on an element from B can be derived when one begins with an automorphism group of just an orientably-regular map with a type-**TW** vertex-quasiprimitive group of orientation-preserving automorphisms.

In what follows we provide constructions of infinite families of non-orientable regular maps and orientably-regular reflexible as well as chiral maps with vertex-quasiprimitive automorphism groups of type **TW**.

Proposition 3.1 *Let a non-Abelian simple group T be the automorphism group of a non-orientable regular map of type (m,m) for some odd $m \geq 3$, such that T has a partial presentation $\langle x,y,z \mid x^2, y^2, z^2, (xy)^2, (yz)^m, (zx)^m, \ldots\rangle$. Assume that T admits an outer automorphism η that interchanges x with y and fixes z. Let $P = \langle \lambda, \rho \mid \lambda^2, \rho^{3\ell}, (\rho\lambda)^2\rangle$ be a dihedral group of order 6ℓ with a dihedral subgroup $Q = \langle \lambda, \rho^3\rangle$ of order 2ℓ for some $\ell \geq 1$. Further, let $\varphi:\ Q \to \mathrm{Aut}(T)$ be a group homomorphism given by $\varphi(\rho^3) = id$ and $\varphi(\lambda) = \eta$, and let $B \rtimes P = T\ \mathrm{twr}_\varphi\ P$ be the corresponding twisted wreath product. Let $b \in B$ be defined by $b(1) = z$, $b(\rho) = x$ and $b(\rho^2) = y$.*

Then, $\mathrm{Map}(B \rtimes P; b, \lambda, \lambda\rho)$ *is a non-orientable regular map of type $(3\ell, 4m)$ with a quasiprimitive automorphism group of type* **TW** *in its action on vertices.*

Proof. For the sake of bookkeeping let us recall that, with P, Q and φ as in the statement, we have $B = \{f:\ P \to T;\ f(\alpha\beta) = f(\alpha)^{\varphi(\beta)}\}$ for every $\alpha \in P$ and $\beta \in Q$. In particular, for every i mod ℓ and every $j \in \{0,1,2\}$, each $f \in B$ satisfies $f(\rho^{3i+j}) = f(\rho^j)$ and $f(\rho^{3i+j}\lambda) = f(\rho^j)^\eta$. Thus, $B \cong T^3$ since every $f \in B$ is determined by its values at 1, ρ and ρ^2, and the values $f(\rho^{3i+j}\lambda)$ as 'words' over the alphabet $\{x,y,z\}$ are obtained from the values $f(\rho^j)$ by interchanging x with y and keeping z unchanged, since this is how the outer automorphism η is assumed to act.

Let $b \in B$ be as in the statement, defined by $b(1) = z$, $b(\rho) = x$ and $b(\rho^2) = y$, and let $G = \langle b, \lambda, \rho\lambda\rangle = \langle b, \lambda, \rho\rangle$. Clearly, G is a subgroup of $B \rtimes P$, and our next aim is to show that G is *equal* to $B \rtimes P$. To do so it is clearly sufficient to prove that B is a subgroup of G; recall that B is a group under pointwise multiplication of functions.

We begin with calculating values (at $1, \rho, \rho^2 \in P$) of a number of products of elements of the form $\alpha^{-1}b \cdot \alpha = b^\alpha \in G$ for $\alpha \in P$, recalling that $b^\alpha(\omega) = b(\alpha\omega)$ for every $\omega \in P$. This way, for $c = b^\rho b^{\rho^2} \in G$ we successively obtain $c(1) = b^\rho b^{\rho^2}(1) = b^\rho(1) b^{\rho^2}(1) = b(\rho)b(\rho^2) = xy$, and, similarly, $c(\rho) = b^\rho b^{\rho^2}(\rho) = b(\rho^2)b(\rho^3) = b(\rho^2)b(1) = yz$ and $c(\rho^2) = b^\rho b^{\rho^2}(\rho^2) = zx$. It follows that for the element $c^m \in G$ we have $c^m(\rho) = (yz)^m = 1$ and $c^m(\rho^2) = (zx)^m = 1$, but $c^m(1) = (xy)^m = xy \neq 1$ as m is odd and xy is assumed to have order 2 in T.

The non-orientability assumption of $\mathrm{Map}(T; x, y, z)$ implies that $T\langle x,y,z\rangle = \langle xy, yz\rangle$. Observing that $xy = c(1)$ and $yz = c(\rho) = c^\rho(1)$, this means that for every $t \in T$ there is a product $a = a_t \in G$ formed by elements $c, c^\rho \in G$ such that $a(1) = t$. But then, $a^{-1}c^m a(1) = t^{-1}(xy)t$ and $a^{-1}c^m a(\rho) = a^{-1}c^m a(\rho^2) = 1$. It follows that G contains a subgroup B_1 consisting of products of conjugates $a^{-1}c^m a \in G \cap B$ with value $1 \in T$ at $\rho, \rho^2 \in P$ and with value $t^{-1}(xy)t \in T$ at $1 \in P$, for every $t \in T$. If $\pi:\ G \cap B \to T$ is the projection epimorphism onto the first coordinate, it follows that $\pi(B_1)$ is a normal subgroup of T. Since B_1 is non-trivial and T is simple, we conclude that $\pi(B_1) = T$. In other words, $B_1 < G$ contains functions with value 1 at $\rho, \rho^2 \in P$ but with arbitrary values from T at $1 \in P$. It is now obvious that letting $B_2 = \{f^{\rho^2};\ f \in B_1\}$ and

$B_3 = \{f^\rho;\ f \in B_1\}$, elements of $B_2 < G$ can have any value from T at ρ but have value $1 \in T$ at $1, \rho^2 \in P$, and elements of $B_3 < G$ assume all values of T at ρ^2 but have value $1 \in T$ at $1, \rho \in P$. This implies that the set $B_1 \cup B_2 \cup B_3$ under pointwise multiplication generates all of B. Thus, B is a subgroup of G, and hence G is equal to $B \rtimes P$.

Clearly, $b \in G$ is an involution. By the definition of the homomorphism $\varphi :\ P \to \mathrm{Aut}(T)$ one has $b^\lambda(1) = b(\lambda) = b(1)^\eta = z^\eta = z = b(1)$, $b^\lambda(\rho) = b(\lambda\rho) = b(\rho^{-1}\lambda) = b(\rho^2\lambda) = b(\rho^2)^\eta = y^\eta = x = b(\rho)$, and $b^\lambda(\rho^2) = b(\lambda\rho^2) = b(\rho\lambda) = b(\rho)^\eta = x^\eta = y = b(\rho^2)$. This shows that the values of b and b^λ agree at $1, \rho, \rho^2$ and so $b = b^\lambda$. It follows that the involutions $b, \lambda \in G$ commute, as $b\lambda = b \cdot \lambda = (1, b)(\lambda, 1) = (\lambda, b^\lambda) = \lambda b^\lambda = \lambda b$. Together with the conclusion from the previous paragraph this shows that $M = \mathrm{Map}(B \rtimes P; b, \lambda, \rho\lambda)$ is a well-defined regular map.

It remains to show that the map M is non-orientable, which is equivalent to proving that $b \in H$ where $H = \langle \lambda b, \rho \rangle$. We begin with observing that the element $c = b^\rho b^{\rho^2}$ which we used before belongs to H, because a calculation in the semi-direct product G with the help of $b = \lambda b \cdot \lambda$ shows that $c = b^\rho b^{\rho^2} = (\rho^{-1} b \cdot \rho)(\rho^{-2} b \cdot \rho^2) = \rho^{-1}\lambda b \cdot \lambda\rho^{-1} b \cdot \rho^2 = \rho^{-1}\lambda b \cdot \rho\lambda b \cdot \rho^2 \in H$. This also implies that $c^\rho \in H$. Now, the conclusion of the paragraph in which we showed that $B < G$ implies that there is a product d_z of conjugates of the form $a^{-1} c^m a \in B$, where the elements a themselves are products of $c, c^\rho \in H$, such that $d_z(1) = z$ and $d_z(\rho) = d_z(\rho^2) = 1$; observe that $d_z \in H$. Applying conjugation by ρ^2 and ρ as in the description of B_2 and B_3 we similarly conclude that there are products d_x and d_y, respectively, of conjugates of the form $(a^{-1} c^m a)^{\rho^2}$ and $(a^{-1} c^m a)^\rho$, with a standing for products of $c^{\rho^2}, c \in H$ and $c^\rho, c^{\rho^2} \in H$, such that $d_x(\rho) = x$, $d_x(1) = d_x(\rho^2) = 1$, and $d_y(\rho^2) = y$, $d_y(1) = d_y(\rho) = 1$; again, $d_y, d_z \in H$. But this implies that $b = d_x d_y d_z$ since b and the product on the right-hand side agree in the values at 1, ρ and ρ^2, so that $b \in H$ as $d_x, d_y, d_z \in H$.

It follows that M is a non-orientable map. Its type is given by the orders of ρ and $\rho\lambda b$ in G. The order of ρ is 3ℓ by the definition of P, and since the function $(\rho\lambda b)^2 = (\rho\lambda)^2 b^{\rho\lambda} b = b^{\lambda\rho^{-1}} b = b^{\rho^{-1}} b = b^{\rho^2} b$ maps $1, \rho, \rho^2 \in P$ onto the elements $yz, zx, xy \in T$ of order m, m and 2, respectively (with m odd), it follows that the order of $\rho\lambda b$ is $4m$. Thus, M is of type $(3\ell, 4m)$, as claimed. Finally, since $G = BP = B\langle \lambda, \rho \rangle$ and the dihedral group $P = \langle \lambda, \rho \rangle$ is the stablizer of a vertex of M, the vertex set of M may be identified with B and so $G = \mathrm{Aut}(M)$ is quasiprimitive, of type **TW**, on the vertex set of M. This completes the proof. □

Our next example shows that there are infinitely many instances of non-orientable regular maps as in the above Proposition.

Example 3.2 (Non-orientable regular maps of type TW.)

Let $T = \mathrm{PSL}(2, p)$ for an arbitrary odd prime p. By Propositions 2.1 and 3.1 of [6], T is the automorphism group of a self-dual non-orientable regular map. Translated into the language of groups and giving more details, the group $T \cong \mathrm{PSL}(2, p)$ admits a presentation of the form $T = \langle x, y, z \mid x^2, y^2, z^2, (xy)^2, (yz)^p, (zx)^p, \ldots \rangle$. Moreover, based on Proposition 3.1 of [6] it can be checked that if -2 is not a square mod p, that is, if $p \equiv 5$ or $7 \bmod 8$, then T admits an *outer* automorphism fixing z and interchanging x with y (we note that this automorphism induces a self-duality mapping of the regular map $\mathrm{Map}(T; x, y, z)$ onto itself). The above Proposition then implies that for every prime p congruent to 5 or 7 mod 8 and for every odd $m \geq 3$ and an arbitrary $\ell \geq 1$ there exists a non-orientable regular map of type $(3\ell, 4m)$ whose automorphism group, regarded as a permutation group on the vertex set of the map, is quasiprimitive and of type **TW**. Note that the vertex set of this map may be identified with the group $(\mathrm{PSL}(2, p))^3$. □

We now prove an auxiliary result about automorphisms of a wreath product, which may be folklore for specialists. Let T be a (not necessarily simple) group and let T^k denote the direct product of k copies of T. We will identify elements of T^k with functions $f : \{1, 2, \ldots, k\} \to T$ to facilitate later applications to twisted wreath products. Consider the natural action of S_k on T^k by permutation of coordinates, that is, for every permutation $\mu \in S_k$ we define an (obvious) automorphism of T^k given by $f \mapsto f^\mu$, where $f^\mu(i) = f(\mu(i))$ for every $i \in \{1, 2, \ldots, k\}$. Further, let J be a subgroup of S_k and let $\theta : J \to \mathrm{Aut(T^k)}$ be the homomorphism given by the restriction of the above action to J. Let $H = T \,\mathrm{wr}_\theta J = T^k \rtimes J$ be the wreath product of T and J with respect to θ. Again, in Suzuki'z notation [30], elements of H have the form αf for $\alpha \in J$ and $f \in T^k$, with multiplication $\alpha f \cdot \beta g = \alpha\beta f^\beta g$. Moreover, let $A = (A_i)_{i=1}^k$ be an automorphism of T^k acting coordinatewise, so that for every $f \in T^k$ we have $(Af)(i) = A_i(f(i))$ for each $i \in \{1, 2, \ldots, k\}$. If A is constant on orbits of a permutation $\beta \in S_k$, that is, if $(A_i)_{i=1}^k = A = A^\beta = (A_{\beta(i)})_{i=1}^k$, then $(Af)^\beta = A^\beta(f^\beta) = A(f^\beta)$ for every $f \in T^k$. As usual, for $\beta, \mu \in S_k$ by β^μ we denote the conjugate $\mu^{-1}\beta\mu$.

Lemma 3.3 *Let $H = T \,\mathrm{wr}_\theta J = T^k \rtimes J$ be the wreath product of T and J with respect to a homomorphism $\theta : J \to \mathrm{Aut}(T^k)$ for a subgroup $J < S_k$ as described above. Then, for every permutation μ from the normalizer of J in S_k and for every automorphism $A = (A_i)_{i=1}^k$ of T^k that is constant on orbits of the permutation group J, the mapping $\Phi : H \to H$ given by $\Phi(\alpha f) = \alpha^\mu A(f^\mu)$ is an automorphism of H.*

Proof. Using $\mu\beta^\mu = \beta\mu$ together with the calculation rules described before and the assumptions that A is constant on orbits of J and $\beta^\mu \in J$ for every $\beta \in J$ we successively obtain $\Phi(\alpha f)\cdot\Phi(\beta g) = \alpha^\mu A(f^\mu)\cdot\beta^\mu A(g^\mu) = \alpha^\mu\beta^\mu(A(f^\mu))^{\beta^\mu}A(g^\mu) = \alpha^\mu\beta^\mu A^{\beta^\mu}(f^{\mu\beta^\mu})A(g^\mu) = \alpha^\mu\beta^\mu A(f^{\beta\mu})A(g^\mu) = (\alpha\beta)^\mu A((f^\beta g)^\mu) = \Phi(\alpha\beta f^\beta g) = \Phi(\alpha f \cdot \beta g)$. This shows that the mapping $\Phi : H \to H$ is a group homomorphism and since its kernel is trivial by inspection, Φ is an automorphism of H. □

By a *word* w over an alphabet $\mathcal{A}$ we mean any element of the free group over $\mathcal{A}$, that is, an arbitrary product $w = z_1 z_2 \ldots z_j$ such that $z_i \in \mathcal{A}$ for every $i \in \{1, 2, \ldots, j\}$. If $\mathcal{A}$ has n elements, we will simply say that w is a word in n variables without any reference to the alphabet.

Proposition 3.4 *Let T be a non-Abelian simple group with a generating set $\{x_1, \ldots, x_k\}$ with the property that there is a word w in two variables such that $w(x_{i-1}, x_i) = 1 \in T$ for* *every $i \in \{2, 3. \ldots, k\}$ but $w(x_k^{-1}, x_1)$ is a non-identity element of T. Further, let T admit an outer automorphism η inverting x_i for every $i \in \{1, 2, \ldots, k\}$. Let $P = \langle \rho \rangle \cong \mathbb{Z}_{2k}$, let $Q = \langle \rho^k \rangle \cong \mathbb{Z}_2$, and let $\varphi : Q \to \mathrm{Aut}(T)$ be a homomorphism given by $\varphi(\rho^k) = \eta$. As before, let $B \rtimes P = T \operatorname{twr}_\varphi P$ be the corresponding twisted wreath product and let $b \in B$ be defined by letting $b(\rho^{i-1}) = x_i$ for $i \in \{1, 2, \ldots, k\}$. Finally, let ℓ be the least common multiple of the orders of the products $w_1 = \Pi_{i=1}^k (y_{2i-1}^{-1} y_{2i})$ and $w_2 = \Pi_{i=1}^k (y_{2i-1} y_{2i}^{-1})$ in T, where $y_j = x_j$ for $j \in \{1, 2, \ldots, k\}$ and $y_j = x_{j-k}^{-1}$ for $j \in \{k+1, \ldots, 2k\}$.*

Then, $\mathrm{Map}(B \rtimes P; \rho, \rho^k b)$ *is an orientably-regular map of type $(2k, 2k\ell)$, with a quasi-primitive automorphism group of type* **TW** *in its action on vertices.*

Proof. In the light of details explained in the proof of Proposition 3.1 we will be more concise. The key points are to show that the twisted wreath product $B \rtimes P = T \operatorname{twr}_\varphi P$ with T, P, φ and B as introduced in the statement is indeed generated by the elements ρ, of order $2k$, and $\rho^k b$, of order 2. Now, $\rho^k b$ is an involution because, for every $\alpha \in P$ one has $b^{\rho^k}(\alpha) = b(\rho^k \alpha) = b(\alpha \rho^k) = b(\alpha)^\eta = b(\alpha)^{-1}$, which means that $b^{\rho^k} = b^{-1}$, and then $\rho^k b \cdot \rho^k b = \rho^{2k} b^{\rho^k} b = \rho^0 b^{-1} b = 1$.

To prove that the group $G = B \rtimes P = T \operatorname{twr}_\varphi P$ is generated by ρ and $\rho^k b$ it is clearly sufficient to show that G contains $B \cong T^k$. By our assumption on the generating set $X = \{x_1, \ldots, x_k\}$ of T, there is a word w over a two-letter alphabet such that $w(x_{i-1}, x_i) = 1 \in T$ for $i \in \{2, \ldots, k\}$ but $w(x_k^{-1}, x_1) = t \in T$ for some $t \neq 1$. Recalling that $x_i = b(\rho^{i-1})$, $1 \leq i \leq k$, we let $a = b^{\rho^{-1}} = \rho b \cdot \rho^{-1}$, so that $a \in G \cap B$ with $a(\rho^{i-1}) = x_{i-1}$ for $i \in \{2, \ldots, k\}$ and $a(1) = b(\rho^{2k-1}) = b(\rho^{k-1})^{-1} = x_k^{-1}$. In the group B we have coordinatewise multiplication, and so the product $w(a, b) \in G \cap B$ evaluated at ρ^{i-1} for $i \in \{2, \ldots, k\}$ gives $w(a, b)(\rho^{i-1}) = w(a(\rho^{i-1}), b(\rho^{i-1})) = w(x_{i-1}, x_i) = 1 \in T$, while $w(a, b)(1) = w(a(1), b(1)) = w(x_k^{-1}, x_1) = t \in T$, $t \neq 1$.

Our assumptions thus imply that the function $w(a, b) \in G \cap B$ has non-identity value $t \in T$ at $1 \in P$ while it has the value 1 for every $\rho^i \in P$ for $i \in \{1, 2, \ldots, k-1\}$. Since X generates T, for every $s \in T$ there is a product $c \in B \cap G$ formed by elements from the set $\{b, b^\rho, \ldots, b^{\rho^{k-1}}\}$ such that $c(1) = s$. But then, as in the proof of Proposition 3.1, simplicity of T implies that the values of products of functions of the form $c^{-1} w(a, b) c \in G \cap B$ at $1 \in P$ cover all of T while having the value $1 \in T$ at the remaining points ρ^i, $1 \leq i \leq k-1$. Applying conjugation by powers of ρ one similarly obtains functions in $G \cap B$ with arbitrary values from T at ρ^i for any fixed $i \in \{1, 2, \ldots, k-1\}$ while having the value $1 \in T$ when evaluated at any other power ρ^j for $j \in \{0, 1, \ldots, k-1\}$, $j \neq i$. This implies that G contains all of $B \cong T^k$, and hence $\langle \rho, \rho^k b \rangle = B \rtimes P$.

It follows that $M = \mathrm{Map}(B \rtimes P; \rho, \rho^k b)$ is an orientably-regular map. Its vertex degree is given by the order of ρ, which is $2k$; observe that we have $k \geq 2$ because of our assumptions on T. The face length of M is the order of the product $\rho \cdot \rho^k b = \rho^{k+1} b$. Since $\rho^{k+1} b \rho^{k+1} b = \rho^2 b^{\rho \eta} b$ and the actions of ρ and η commute in our case, the smallest power of $\rho^{k+1} b$ with first coordinate equal to $1 \in P$ is $2k$. For the element $c = (\rho^{k+1} b)^{2k} \in B$ we then obtain $c = (\rho^2 b^{\rho \eta} b)^k = (b^{\rho^{2k-1} \eta} b^{\rho^{2k-2}}) \ldots (b^{\rho^3 \eta} b^{\rho^2})(b^{\rho \eta} b)$. A calculation shows that $c(1) = w_1^{-1}$ and $c(\rho) = x_1^{-1} w_2^{-1} x_1$, where w_1 and w_2 are as given in the statement of our proposition, and it is easy to see that the values $c(\rho^i)$ for even and odd powers

$i \in \{2, \ldots, k-1\}$ are conjugates of $c(1)$ and $c(\rho)$, respectively. It follows that the order of $\rho^{k+1}b$, that is, the face length of M, is equal to $2k$ times the least common multiple of the orders of w_1 and w_2 in T, as claimed. Again, since the group G has the form $G = BP = B\langle\rho\rangle$ and the cyclic group $P = \langle\rho\rangle$ is a vertex stabilizer, the vertex set of M may be identified with B and $G = \mathrm{Aut}(M)$ is a quasiprimitive permutation group of type **TW** on vertices of M. $\square$

We note that the condition on the word w in the statement of Proposition 3.4 was chosen to be simple enough and still allowing for interesting applications in what follows. The role of the condition is just to guarantee generation of $B \cong T^k$ by ρ-conjugates of b. One can modify the condition by replacing w by a word in more than two variables, or by several such words, satisfying suitable assumptions serving the purpose just described.

Example 3.5 (Reflexible orientably-regular maps of type TW.)
Let $k \geq 2$ and let $T = A_{k+2}$ be the alternating group on the set $\{u, v, 1, 2, \ldots, k\}$ of $k+2$ elements. For $i \in \{1, 2, \ldots, k\}$ let $x_i = (u, v, i) \in T$; the set $X = \{x_1, \ldots, x_k\}$ is known to generate A_{k+2}. Reading compositions of permutations from the left to the right and letting $w(x, y) = (xy)^2$, for $i \in \{2, \ldots, k\}$ we obtain $w(x_{i-1}, x_i) = ((u, v, i-1)(u, v, i))^2 = ((u, i)(v, i-1))^2 = id$, while $w(x_k^{-1}, x_1) = ((v, u, k)(u, v, 1))^2 = (u, k, 1)^2 = (u, 1, k)$, a non-identity element of T. Further, conjugation by the transposition (u, v) induces an outer automorphism η of A_{k+2} inverting every element of X. Let $P = \langle\rho\rangle \cong \mathbb{Z}_{2k}$ and let $Q = \langle\rho^k\rangle$. By Proposition 3.4, the map $M = \mathrm{Map}(B \rtimes P; \rho, \rho^k b)$ with $b(\rho^{i-1}) = x_i = (u, v, i)$ for $i \in \{1, 2, \ldots, k\}$ is orientably-regular, of degree $2k$. A calculation shows that the elements w_1 and w_2 have both orders $k+2$ if k is odd and $(k+2)/2$ if k is even, implying that the face length of M is $2k(k+2)$ if k is odd and $k(k+2)$ if k is even. The automorphism group of the map M is quasiprimitive, of type **TW**, on its vertex set (that may be identified with the group $B \cong T^k = A_{k+2}^k$).

In addition, we show that our map M is reflexible, which amounts to showing that there is an automorphism of $B \rtimes P$ inverting ρ and preserving $\rho^k b$. To be able to apply Lemma 3.3 for this purpose we will identify the generator ρ of $P \cong \mathbb{Z}_{2k}$ simply with the cyclic permutation $(0, 1, 2, \ldots, 2k-1)$ of the exponents at ρ. Let μ be an involutory permutation of the same set of exponents given by $i \mapsto 2k - i$ for $i \in \{0, 1, \ldots, k\}$; obviously $\rho^\mu = \mu\rho\mu = \rho^{-1}$ and $P\langle\mu\rangle$ is a dihedral group of order $4k$. Consider the automorphism of $T = A_{k+2}$ given by conjugation by the permutation $\omega = (u, v)\Pi(i+1, k-i+1)$ of the set $\{u, v, 1, 2, \ldots, k\}$, where the product extends over all i such that $1 \leq i \leq \lfloor (k-1)/2 \rfloor$. A calculation shows that the image $b(\rho^i)^\omega = (u, v, i+1)^\omega = \omega(u, v, i+1)\omega$ is equal to $(v, u, k-i+1) = b((\rho^{k-i})^{-1}$ for $i \in \{1, \ldots, k-1\}$ and to $(v, u, 1) = b(1)^{-1}$ for $i = 0$. This implies that $b(\rho^i)^\omega = b((\rho^{k-i})^{-1} = b(\rho^{k-i})^\eta = b(\rho^{k-i}\rho^k) = b(\rho^{2k-i}) = b^\mu(\rho^i)$ for $i \in \{0, 1, \ldots, k-1\}$. It follows that for the automorphism A of B given by $A(f(\alpha)) = f(\alpha)^\omega$ for every $\alpha \in P$ we have $A(b) = b^\mu$, and also $b = A(b^\mu)$ as A is involutory. Since A is constant on the (single) orbit of P and μ normalizes P, by Lemma 3.3 the group $B \rtimes P$ has an automorphism Φ given by $\Phi(\alpha f) = \alpha^\mu A(f^\mu)$ for every $\alpha \in P$ and $f \in B$. Clearly, $\Phi(\rho) = \rho^\mu = \rho^{-1}$, and using $b = A(b^\mu)$ we obtain $\Phi(\rho^k b) = (\rho^k)^\mu A(b^\mu) = \rho^k b$. Thus, Φ is an orientation-reversing automorphism of our map M, establishing its reflexibility. $\square$

Example 3.6 (Chiral orientably-regular maps of type TW.)
Let $k \geq 3$ and let $T = \mathrm{PSL}(2,p)$ for an odd prime $p \equiv 3 \bmod 4$. By [6] the group T can be generated by an element x_1 of order 2 and an element x_2 of order p such that x_1x_2 has again order p. To be explicit, we may take

$$x_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } x_2 = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}, \text{ with } x_1x_2 = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}.$$

Further, let x_i for $i \in \{3,4,\ldots,k\}$ be arbitrary elements of T of the form

$$y_i = \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix} \quad \text{or} \quad z_i = \begin{pmatrix} 0 & -1 \\ 1 & \xi{+}\xi^{-1} \end{pmatrix}$$

where $\xi = \xi_i$ is an arbitrary primitive $2m_i$-th root of unity for some *odd* divisor m_i of $(p-1)/2$ if $x_i = y_i$, or of $(p+1)/2$ if $x_i = z_i$; note that this makes x_i an element of order m_i in T. Observe that the group $T = \mathrm{PSL}(2,p)$ admits an outer automorphism η, given by conjugation by the matrix with 0's and 1's as diagonal and off-diagonal elements, that inverts every x_i $(1 \leq i \leq k)$.

We may assume that the word w from Proposition 3.4 depends just on the second variable and has the form $w(y) = y^n$ where n is the least common multiple of the orders of x_i for $i \in \{2,3,\ldots,k\}$. Observe that n is odd in our case, and so $w(x_i) = 1$ for every $i \in \{2,3,\ldots,k\}$ but $w(x_1) = x_1^n = x_1 \neq 1$. Let $P = \langle \rho \rangle \cong \mathbb{Z}_{2k}$ and $Q = \langle \rho^k \rangle$ as before. By Proposition 3.4, the map $M = \mathrm{Map}(B \rtimes P; \rho, \rho^k b)$ with $b(\rho^{i-1}) = x_i$ for $i \in \{1,2,\ldots,k\}$ is orientably-regular, of degree $2k$, and with $\mathrm{Aut}(M)$ being a quasiprimitive permutation group of type **TW** on the vertex set (which, again, may be identified with the group $B \cong T^k = (\mathrm{PSL}(2,p))^k$). The face length of M cannot be determined more explicitly at this level of generality because of a rather arbitrary choice of x_i for $3 \leq i \leq k-1$.

It remains to prove that every map M constructed in this example is chiral. We will begin with a few preliminary observations. Let $\ell = 2n/p$ be the least common multiple of the orders of x_1 and x_i for $i \in \{3,4,\ldots,k\}$; the choice of our generators implies that ℓ is relatively prime to p and so there exists a multiple m of ℓ such that $m \equiv 1 \bmod p$. Observe that $b^n(1) = b^n(\rho^k) = x_1$ while all the remaining values of b^n are 1. Similarly, one obtains $(b^\rho)^m(1) = (b(\rho))^m = x_2$ and $(b^\rho)^m(\rho^k) = (b(\rho)^{-1})^m = x_2^{-1}$, with all the remaining values of $(b^\rho)^m$ being equal to 1, and also $(b^{\rho^{-1}})^m(\rho^2) = (b(\rho))^m = x_2$ and $(b^{\rho^{-1}})^m(\rho^{k+2}) = (b(\rho)^{-1})^m = x_2^{-1}$ while the remaining values of $(b^{\rho^{-1}})^m$ are 1. It follows that $b^n(b^\rho)^m(1) = x_1x_2$ and $b^n(b^\rho)^m(\rho^k) = x_1x_2^{-1}$, with the remaining values of $b^n(b^\rho)^m$ being 1. On the other hand, recalling that $k \geq 3$, we have $b^n(b^{\rho^{-1}})^m(1) = (b^{\rho^{-1}})^m(\rho^k) = x_1$ and $(b^{\rho^{-1}})^m(\rho^2) = (b^{\rho^{-1}})^m(\rho^{k+2}) = x_2^{-1}$, while all the remaining values of $(b^{\rho^{-1}})^m$ are 1. This implies that the order of $b^n(b^\rho)^m$ is equal to the order of x_1x_2 (and equal to the order of $x_1x_2^{-1}$ as x_1 is an involution), which is p, but the order of $b^n(b^{\rho^{-1}})^m$ is equal to the least common multiple of the orders of x_1 and x_2, which is $2p$.

Suppose now that M is reflexible and let Φ be an automorphism of $B \rtimes P$ preserving $\rho^k b$ and inverting ρ. It follows that Φ must preserve b, and hence b^n, and send b^ρ onto $b^{\rho^{-1}}$, and hence also $(b^\rho)^m$ onto $(b^{\rho^{-1}})^m$. Consequently, Φ must send the product $b^n(b^\rho)^m$

onto $b^n(b^{\rho^{-1}})^m$. However, the calculations from the previous paragraph imply that the two products have different orders in T, a contradiction. We conclude that all our maps M in this example are chiral.

Finally, observe that if k is odd and $x_3 = \ldots = x_k = 1$ in the above construction, then for the elements w_1 and w_2 from the statement of Proposition 3.4 we have $w_1 = (x_1^{-1}x_2)^2$ and $w_2 = (x_1x_2^{-1})^2$, both of order p. Consequently, in such a special case we have a chiral orientably-regular map of degree $2k$ and face length $2kp$ for every odd $k \geq 3$ and every prime $p \equiv 3 \bmod 4$, with a quasiprimitive automorphism group of type **TW** on the vertex set. □

4 Quasiprimitive regular maps of type PA

We begin with a description of quasiprimitive groups of product action type (**PA**) given in [26]. Let G be a quasiprimitive permutation group of type **PA** on a set Ω and let $N = T^k$ be the socle of G, where T is a non-Abelian finite simple group and $k \geq 2$. Then, the group G satisfies $N \leq G \leq H \wr S_k \leq \mathrm{Sym}(\Delta) \wr S_k$, where $H \leq \mathrm{Sym}(\Delta)$ is an almost simple quasiprimitive permutation group on some set Δ with socle T.

Let $\pi = (1, 2, \ldots, k)$ and let $J = \langle\pi\rangle$. As in Section 3, we will identify elements of $N = T^k$ with functions $f : \{1, 2, \ldots, k\} \to T$, and for every permutation $\mu \in J$ we define $f^\mu(i) = f(\mu(i))$, for each $i \in \{1, 2, \ldots, k\}$. The 'smallest' choice for a group of type **PA** is to take $G = T \wr J := N \rtimes J$. Elements of G have the form λf with $\lambda \in J$ and $f \in N$, with multiplication $\mu f \cdot \nu g = \mu\nu f^\nu g$.

Take a point $\alpha \in \Omega$. Because N acts transitively on Ω, one may identify Ω with the cosets of the stabilizer N_α in N and assume that $\Omega = \{N_\alpha f \mid f \in N\}$. Then, the result of an action of a group element $\nu g \in G$ on an element $N_\alpha f \in \Omega$ is given by $N_\alpha f \cdot \nu g = N_\alpha f^\nu g$.

Let R be a point stabilizer of the action of T on Δ; we may assume that $\Delta = \{Rt \mid t \in T\}$. By the conditions in [26], the group R cannot be a maximal subgroup of T because T is assumed (by a further requirement for membership in the **PA** family) to be a quasiprimitive but not a primitive permutation group on the set Δ. Clearly, R^k is a subgroup of N. Moreover, there is a G-invariant partition of Ω induced by the block $\{N_\alpha h \mid h \in R^k\}$ and its G-translates.

In [17] the authors divide the quasiprimitive groups of **PA** type into three subfamilies: straight diagonal, twisted diagonal, and non-diagonal. Using the above notation, the group G is *diagonal* if $N_\alpha \cong R$. In this case, one has $N_\alpha = \{f \mid f(i) = r^{\varphi_i}, r \in R, 1 \leq i \leq k\}$, where $\varphi_i, 1 \leq i \leq k$, are automorphisms of R. If every automorphism $\varphi_i, 1 \leq i \leq k$, extends to an automorphism of T, then G is called *straight diagonal*. Otherwise, if at least one of these φ_i's, $1 \leq i \leq k$, cannot be extended to an automorphism of T, the group G (and its action) is called *twisted diagonal*. Finally, if N_α is not isomorphic to R but is a subdirect product subgroup of R^k, then G is called *non-diagonal*.

In what follows, we will only consider the *very special* case where $\pi \in G_\alpha$. While this assumption simplifies our arguments, it still allows us to construct the desired examples.

Lemma 4.1 *If G is straight diagonal and $\pi \in G_\alpha$, then the automorphisms $\varphi_i, 1 \leq i \leq k$, of R appearing in N_α satisfy the identities*

$$\varphi_{\nu(1)}\varphi_1^{-1} = \varphi_{\nu(2)}\varphi_2^{-1} = \ldots = \varphi_{\nu(k)}\varphi_k^{-1}$$

for every $\nu \in J = \langle \pi \rangle$.

Proof. As $\pi \in G_\alpha$, under the identification of α with N_α, each element $\nu \in J$ fixes N_α. This means that for each $f \in N_\alpha$ we have $f^\nu \in N_\alpha$. Thus, there exist two elements of R, say r and $\tilde{r}$, such that $f(i) = r^{\varphi_i}$ and $f^\nu(i) = \tilde{r}^{\varphi_i}$ for every $1 \leq i \leq k$. Because $f^\nu(i) = f(\nu(i))$, it follows that $r^{\varphi_{\nu(i)}} = \tilde{r}^{\varphi_i}$ for each $1 \leq i \leq k$. This implies that $r^{\varphi_{\nu(i)}\varphi_i^{-1}} = r^{\varphi_{\nu(j)}\varphi_j^{-1}}$ for every $1 \leq i, j \leq k$ and each $r \in R$. So, $\varphi_{\nu(1)}\varphi_1^{-1} = \varphi_{\nu(2)}\varphi_2^{-1} = \ldots = \varphi_{\nu(k)}\varphi_k^{-1}$ for each $\nu \in J = \langle \pi \rangle$. □

As a very special case of Lemma 4.1, we have the following corollary.

Corollary 4.2 *If there is an automorphism φ of R such that $\varphi^k = id_R$, then $N_\alpha = \{f \mid f(i) = r^{\varphi^i}, r \in R, 1 \leq i \leq k\}$ is fixed by J and $G_{N_\alpha} \cong N_\alpha \rtimes J$.*

Proof. By the choice of the automorphisms, $\varphi_i = \varphi^i$. Letting $f \in N_\alpha$, we obtain $f(i) = r^{\varphi^i}, 1 \leq i \leq k$, for some $r \in R$. For each $\pi^\epsilon, 1 \leq \epsilon \leq k$, it follows from $\pi^\epsilon(i) \equiv \epsilon + i \bmod k$ that $f^{\pi^\epsilon}(i) = f(\pi^\epsilon(i)) = f(\epsilon + i) = r^{\varphi^{\epsilon+i}} = (r^{\varphi^\epsilon})^{\varphi^i}$. Thus, $f^{\pi^\epsilon} \in N_\alpha$ and this means that N_α is fixed under the action of $J = \langle \pi \rangle$. Moreover, $\varphi_{\pi^\epsilon(i)}\varphi_i^{-1} = \varphi_{\epsilon+i}\varphi_i^{-1} = \varphi^{\epsilon+i}\varphi^{-i} = \varphi^\epsilon$, independently of i. By Lemma 4.1, the group G is straight diagonal. For $\mu f \in G_{N_\alpha}$, we have $N_\alpha^{\mu f} = N_\alpha f = N_\alpha$, so that $f \in N_\alpha$, and hence $G_{N_\alpha} \cong N_\alpha \rtimes J$. □

It is clear that G_{N_α} in Corollary 4.2 is Abelian if and only if $\varphi = id_R$. In this case, G is a straight diagonal group of **PA** type, and for such groups G we prove the following two results. We continue using the notation introduced in the description of groups of **PA** type. Also, we will assume that T is an orientation-preserving automorphism group of some orientably-regular map.

Proposition 4.3 *Let $T = \langle\, r, \tau \mid r^\ell, \tau^2, \ldots \,\rangle$ and let $R = \langle r \rangle$. If $N_\alpha = \{f \mid f(i) \in R, f(i) = f(j), 1 \leq i, j \leq k\}$ and $(\ell, k) = 1$, then $G_{N_\alpha} \cong \mathbb{Z}_{k\ell}$ and*

$$G = \langle\, \pi f, g \mid f \in N_\alpha, f(i) = r, 1 \leq i \leq k, g \in N, g(1) = \tau, g(i) = 1, 2 \leq i \leq k \rangle.$$

Moreover, the map $\mathcal{M} = \mathrm{Map}(G; \pi f, g)$ is orientably-regular, of degree $k\ell$, with a straight diagonal quasiprimitive orientation-preserving automorphism group G of type **PA** *acting on its vertices.*

Proof. Because $(\ell, k) = 1$, the order of πf is equal to $k\ell$, the element $(\pi f)^\ell = \pi^\ell$ generates J, and $(\pi f)^k = f^k$ generates N_α. To show that G is generated by πf and g, we only need to show that N is generated by πf and g. Note that, by the definition of f, its inverse $f^{-1} \in N$ satisfies $f^{-1}(i) = r^{-1}, 1 \leq i \leq k$.

For any pair of positive integers j and ϵ, we have $(f^{-j} g^{\pi^\epsilon} f^j)(i) = f^{-j}(i) g^{\pi^\epsilon}(i) f^j(i) = r^{-j} g(\epsilon + i) r^j$, where $\epsilon + i$ is calculated mod k. So, if $\epsilon + i = 1$ mod k, then $(f^{-j} g^{\pi^\epsilon} f^j)(i) = r^{-j} \tau r^j$, and if $2 \leq \epsilon + i \leq k$ mod k, then $(f^{-j} g^{\pi^\epsilon} f^j)(i) = 1$. Therefore, one may choose an ϵ to obtain a series of elements in N, denoted $h_{i,j}$, such that $h_{i,j}(i) = r^{-j} \tau r^j$ and $h_{i,j}(i') = 1$, where $1 \leq i' \neq i \leq k, 1 \leq j \leq \ell$.

For every $h \in N$, define $h_i, 1 \leq i \leq k$, by letting $h_i(i) = h(i)$ and $h_i(i') = 1$ for $1 \leq i' \neq i \leq k$, so that $h = h_1 h_2 \cdots h_k$. Note that the group generated by the elements $\{r^{-j} \tau r^j\}$ is a normal subgroup of T, and $T = \langle r^{-j} \tau r^j \mid j \geq 1 \rangle$. It follows that h_i is a product of $h_{i,j}$'s for some integers j. Therefore, N is generated by πf and g. It is clear that the order of g is equal to 2 and hence G is the automorphism group of an orientably-regular map $\mathcal{M}$ of degree $k\ell$, as stated. □

Theorem 4.4 *Keeping the assumptions of Proposition 4.3, suppose further that the order of $r\tau$ is m. If $(m, \ell) = 1$, then the orientably-regular map $\mathcal{M} = \mathrm{Map}(G; \pi f, g)$ is reflexible if and only if T is the automorphism group of some reflexible regular map.*

Proof. If T is the automorphism group of a reflexible regular map, then there is a $\varphi \in \mathrm{Aut}(T)$ such that $\varphi(r) = r^{-1}$ and $\varphi(\tau) = \tau$. Define $A = (A_i)_{i=1}^k$ to act coordinatewise on $N = T^k$ by $A_i(h(i)) = \varphi(h(i))$, for every $h \in N$. Clearly, A is constant on the orbits of the permutation group $J = \langle \pi \rangle$. Take a permutation $\mu \in S_k$ defined by $\mu = (1)(2, k)(3, k-1) \cdots (\frac{k+1}{2}, \frac{k+3}{2})$ for odd k, and $\mu = (1)(\frac{k+2}{2})(2, k) \cdots (\frac{k}{2}, \frac{k+4}{2})$ for even k. Then, $\pi^\mu = \mu^{-1} \pi \mu = \pi^{-1}$. Define a mapping Φ on G such that $\Phi(\nu h) = \nu^\mu A(h^\mu)$ for each element $\nu h \in G$. By Lemma 3.3, Φ is an automorphism of G, and, in particular, $\Phi(\pi f) = \pi^{-1} A(f^\mu)$. Note that $A(f^\mu)(i) = A_i(f^\mu(i)) = A_i(f(\mu(i))) = \varphi(r) = r^{-1}$, so $A(f^\mu) = f^{-1}$ and $\Phi(\pi f) = \pi^{-1} f^{-1} = (\pi f)^{-1}$. Similarly, $A(g^\mu)(i) = \varphi(g(\mu(i)))$. If $i = 1$, then $A(g^\mu)(1) = \varphi(g(\mu(1))) = \varphi(g(1)) = \varphi(\tau) = \tau$, and for $2 \leq i \leq k$, we have $\mu(i) \neq 1$ and so $A(g^\mu)(i) = \varphi(g(\mu(i))) = 1$. Therefore, $\Phi(g) = A(g^\mu) = g$, and the map $\mathcal{M} = \mathrm{Map}(G; \pi f, g)$ is reflexible.

Conversely, if G is the automorphism group of a reflexible regular map, then there is an automorphism of G, say Φ, such that Φ inverts the two generators of G, that is, $\Phi(\pi f) = \pi^{-1} f^{-1}$ and $\Phi(g) = g$. By taking the powers of πf, we obtain $\Phi(f^i) = f^{-i}$, for each i. As a result, we have $\Phi(fg) = f^{-1} g$, and so $\Phi((fg)^\ell) = (f^{-1} g)^\ell$. Note that $fg(1) = r\tau$, and $fg(i) = r$ for $2 \leq i \leq k$. Thus, $(fg)^\ell(1) = (r\tau)^\ell$, and $(fg)^\ell(i) = 1$, for $2 \leq i \leq k$. Choose $h \in N$ such that $h(1) = r\tau$ and $h(i) = 1$ for $2 \leq i \leq k$, and also choose $\tilde{h} \in N$ such that $\tilde{h}(1) = r^{-1}\tau$ and $\tilde{h}(i) = 1$ for $2 \leq i \leq k$. As $(\ell, m) = 1$, h is a power of $(fg)^\ell$ and $\tilde{h}$ is the same power of $(f^{-1}g)^\ell$. Therefore, $\Phi(h) = \tilde{h}$. It is obvious that hg and g generate a subgroup of N isomorphic to T. The restricted action of Φ on this subgroup can be viewed as an automorphism of T that inverts r and fixes τ, implying that the map $\mathcal{M} = \mathrm{Map}(T; r, \tau)$ is reflexible. □

Example 4.5 (Reflexible orientably-regular maps of type PA.)

Let $T = A_5$ and $R = \langle(1,2,3,4,5)\rangle = \langle r\rangle$. Then, $A_5 = \langle(1,2,3,4,5),(1,5)(3,4)\rangle = \langle r,\tau\rangle$, and R is not maximal in A_5. For any integer $k \geq 3$ such that $(5,k) = 1$, let $G = A_5 \wr (1,2,\cdots,k) = N \rtimes \pi$. Set $N_\alpha = \{h \mid h(i) = t, 1 \leq i \leq k, t \in R\} \cong R$, $f \in N, f(i) = r, 1 \leq i \leq k$, and $g \in N$, $g(1) = \tau, g(i) = 1$, $2 \leq i \leq k$. According to Proposition 4.3, $\mathrm{Map}(G;\pi f, g)$ is an orientable regular map of degree $5k$.

Note that the conjugation of A_5 by the element $(1,3)(4,5)$ reverses both r and τ, so $\mathrm{Map}(T;r,\tau)$ is reflexible. Moreover, the order of $r\tau$ is 3. So, from the result of Theorem 4.4, $\mathrm{Map}(G;\pi f, g)$ is reflexible. □

Example 4.6 (Chiral orientably-regular maps of type PA.)

It is well-known that for each $q = 2^{2m+1}, m \geq 1$, the Suzuki group of order $q^2(q^2+1)(q-1)$, denoted by $\mathrm{Sz}(q)$, is simple and has the presentation

$$\mathrm{Sz}(q) = \langle r, \tau \mid r^4 = \tau^2 = (r\tau)^5 = \cdots = 1\rangle.$$

Let $R = \langle r\rangle$ be the cyclic subgroup of $\mathrm{Sz}(q)$ generated by r. Then R is not maximal in $\mathrm{Sz}(q)$. Set $G = \mathrm{Sz}(q) \wr (1,2,\ldots,k) = N \rtimes \pi$, and $N_\alpha = \{h \mid h(i) = t, 1 \leq i \leq k, t \in R\} \cong R$. Then G is a quasiprimitive **PA** type group. Assume $(k,4) = 1$. Take $f \in N, f(i) = r, 1 \leq i \leq k$, and $g \in N$, $g(1) = \tau, g(i) = 1$, $2 \leq i \leq k$. According to Proposition 4.3, $\mathrm{Map}(G;\pi f, g)$ is an orientably-regular map of degree $4k$.

Since no automorphism of $\mathrm{Sz}(q)$ can reverse simultaneously both r and τ, and the order of $r\tau$ is 5, using Theorem 4.4 we can conclude that $\mathrm{Map}(G;\pi f, g)$ is chiral orientably-regular. □

Next, assume $N_\alpha = \{f \mid f(i) = r^{\varphi^i}, r \in R, 1 \leq i \leq k\}$, where $\varphi \in \mathrm{Aut}(R)$ but $\varphi \neq id_R$. Then, $G_{N_\alpha} \cong N_\alpha \rtimes J$. For this case, we have the following Theorem 4.8. But before that, we prove Lemma 4.7 which will then be used to prove Theorem 4.8.

Lemma 4.7 *Let T be a simple group, $T = \langle r,\tau\rangle$, $f = (r, r^{-1}) \in T\times T$ and $g = (\tau,\tau^{-1}) \in T \times T$. The product $T \times T$ is generated by f and g if and only if no automorphism of T reverses simultaneously r and τ.*

Proof. Suppose there is an automorphism of T, say φ, that can reverse both r and τ. Then, $r^{-1} = \varphi(r)$ and $\tau^{-1} = \varphi(\tau)$. As a result, for each element $h \in \langle f, g\rangle$, we have $h^{-1} = \varphi(h)$. This means $\langle f, g\rangle \cong T$. So, if $T\times T$ is generated by f and g, no automorphism can reverse both r and τ.

On the other hand, if no automorphism of T can reverse both r and τ, then there is a word $w = \prod_{s,t=1}^{n} r^{i_s}\tau^{j_t} = 1_T$, but $w^- = \prod_{s,t=1}^{n} r^{-i_s}\tau^{-j_t} \neq 1_T$. Thus, all the elements of the form $(w^-, 1_T)$ and $(1_T, w^-)$ belong to $T \times T$. Let

$$W = \{w^- \mid w \text{ is a word over the powers of } r \text{ and } \tau \text{ such that } w = 1_T, w^- \neq 1_T\},$$

and let $K = \langle w^- \mid w^- \in W \rangle$. Then K is a normal subgroup of T and so $K = T$. In fact, for each $w^- \in W$, the relations $rwr^{-1} = 1, r^{-1}w^-r \neq 1_T$ and $\tau w \tau^{-1} = 1, \tau^{-1}w^-\tau \neq 1_T$ imply that K is normal. Therefore, $T \times T$ can be generated by f and g. □

Theorem 4.8 *Let G be a finite quasiprimitive* **PA** *type group acting on a set Ω. Assume $T = \langle r, \tau \mid r^\ell, \tau^2 \rangle$ is a non-Abelian simple group and $R = \langle r \rangle$ is not maximal in T. Take $\alpha \in \Omega$. If $N_\alpha = \{f \mid f(i) = r^{\varphi^i}, r \in R, 1 \leq i \leq k\}$, where $\varphi \in \mathrm{Aut}(R)$ but $\varphi \neq id_R$, then G can be the automorphism group of a regular map only if $k = 2$ and $\varphi(r) = r^{-1}$. That is,*

$$G = (T \times T) \rtimes (1, 2) = N \rtimes \pi.$$

Moreover, in this case, G belongs to the twisted diagonal **PA** *type group.*

Set $f = (r, r^{-1}) \in N_\alpha$. If no automorphism of T can reverse both r and τ, then $\mathrm{Map}(G; \pi f, \pi, g)$ is a nonorientably-regular map and $\mathrm{Map}(G; \pi f, \pi, \pi g)$ is a reflexible orientably-regular map, with $g = (\tau, \tau)$ in both cases. If no automorphism of T can reverse both r and $r\tau r^{-1}\tau$, the map $\mathrm{Map}(G; \pi f, \pi, g)$ is nonorientably-regular, with $g = (\tau, r\tau r^{-1})$, and $\mathrm{Map}(G; \pi f, \pi, \pi\tilde{g})$ is reflexible orientably-regular, with $\tilde{g} = (r\tau, \tau r^{-1})$.

Proof. From the preceding discussion, in this case $G_{N_\alpha} \cong N_\alpha \rtimes J$ with $J = \langle \pi \rangle$. If G can be the automorphism group of a regular map, then G_{N_α} should be isomorphic to a dihedral group. Therefore, $k = 2$, $\pi = (1, 2)$, and $r^\varphi = r^{-1}$. Because T is non-Abelian, φ cannot be extended to an automorphism of T. Therefore, G belongs to the twisted diagonal **PA** type group. More precisely,

$$G = (T \times T) \rtimes \pi := N \rtimes \pi$$

with $G_{N_\alpha} = \{f, \pi f \mid f = (f_1, f_2) \in N, f_1 \in R, f_2 = f_1^{-1}\}$. Without loss of generality, we assume $G = \langle \pi f, \pi, g \mid (\pi f)^2, \pi^2, g^2, \ldots \rangle$, where the first two involutions πf and π come from G_{N_α} and the last involution $g \in G \setminus G_{N_\alpha}$.

Case 1. If $g\pi = \pi g$ but $g\pi f \neq \pi f g$, then $g \in N, g_1 = g_2 = a$ with $ar \neq ra$ or $g = \pi\tilde{g}$ with $\tilde{g} \in N, \tilde{g}_1 = \tilde{g}_2 = a$ and $ar^{-1} \neq ra$ for some involution $a \in T$.

For the first subcase, take $g \in N$ such that $g_1 = g_2 = \tau$ and denote the subgroup of G generated by f and $\pi f g$ by H. The index of H in G is at most 2. Let $h = f(\pi f g)^2$. Then $h_1 = \tau r \tau$ and $h_2 = \tau r^{-1} \tau$. Note that the subgroup generated by r and $\tau r \tau$ is a normal subgroup of T, so they generate T. According to Lemma 4.7, N is generated by f and h. So, H is indeed the whole group G. As a result, $\mathrm{Map}(G; \pi f, \pi, g)$ is a nonorientably-regular map.

For the second subcase, we take $\tilde{g} \in N$ such that $\tilde{g}_1 = \tilde{g}_2 = \tau$ and denote the subgroup of G generated by f and $\tilde{g}$ by H. Clearly, H is equal to N. So, in this case, $\mathrm{Map}(G; \pi f, \pi, \pi\tilde{g})$ is a reflexible orientably-regular map.

Case 2. If $g\pi f = \pi f g$ but $g\pi \neq \pi g$, then $g \in N, g_1 = a$ and $g_2 = rar^{-1}$ for some $a \in T$ satisfying $ar \neq ra, a^2 = 1_T$, or $g = \pi\tilde{g}$, where $\tilde{g} \in N$ such that $\tilde{g}_1 = a$ and $\tilde{g}_2 = r^{-1}ar^{-1}$ for some $a \in T$ satisfying $r^{-1}ar^{-1} = a^{-1} \neq a$.

For the first subcase, we take $g \in N$ such that $g_1 = \tau$ and $g_2 = r\tau r^{-1}$. We denote the subgroup of G generated by f and πg by H. Note that $(\pi g)^2 \in N$ with $(\pi g)_1^2 = r\tau r^{-1}\tau$ and $(\pi g)_2^2 = ((\pi g)^2(1))^{-1}$. Similar to **Case 1**, under the condition of no automorphism of T reversing both r and $r\tau r^{-1}\tau$, we have $H = G$. So, $\mathrm{Map}(G; \pi f, \pi, g)$ is a nonorientably-regular map.

For the second subcase, we take $g = \pi\tilde{g}$ such that $\tilde{g} \in N$ with $\tilde{g}_1 = r\tau$ and $\tilde{g}_2 = \tau r^{-1}$. It is clear that the subgroup generated by f and $\tilde{g}$ is of index two in G. So, $\mathrm{Map}(G; \pi f, \pi, \pi\tilde{g})$ is a reflexible orientably-regular map. $\square$

Example 4.9 Nonorientably-regular maps of type PA.

Let $T = \mathrm{Sz}(q)$, where $q = 2^{2m+1}, m \geq 1$. Then $\mathrm{Sz}(q) = \langle r, \tau \mid r^4 = \tau^2 = (r\tau)^5 = \cdots = 1\rangle$. Let $R = \langle r \rangle$ be the cyclic subgroup of $\mathrm{Sz}(q)$ generated by r. Then R is not maximal in $\mathrm{Sz}(q)$ and $G = (\mathrm{Sz}(q) \times \mathrm{Sz}(q)) \rtimes (1, 2) = N \rtimes \pi$ is a **PA** type group. Set $N_\alpha = \{f = (r, r^{-1}) \mid r \in R\}$, $g = (g_1, g_2) \in N$ with $g_1 = g_2 = \tau$. Then, according to Theorem 4.8, $\mathrm{Map}(G; \pi f, \pi, g)$ is a nonorientably-regular map (for every choice of r). $\square$

The last case of the finite **PA** type group is called **nondiagonal**, in which case N_α is not isomorphic to R but is a subdirect product subgroup of R^k.

Lemma 4.10 *If N_α is a subdirect product subgroup of R^k, then N_α cannot be cyclic.*

Proof. Suppose N_α is cyclic and $N_\alpha = \langle f \rangle$ with $f(i) = r_i, r_i \in R, 1 \leq i \leq k$. Since N_α is a subdirect product subgroup of R^k, all projections on R are surjective. It follows that the order of $r_i, 1 \leq i \leq k$, equals to the cardinality of R. So, R is cyclic. As a result, there are automorphisms $\varphi_i, 2 \leq i \leq k$, such that $r_i = r_1^{\varphi_i}$. But, then the group N_α would be diagonal, a contradiction. $\square$

Theorem 4.11 *No nondiagonal* **PA** *type group can be the automorphism group of a (orientably) regular map.*

Proof. Because $G_\alpha = N_\alpha \rtimes (1, 2, \cdots, k)$, it follows that if G_α is cyclic or dihedral then N_α should be cyclic. This is impossible according to Lemma 4.10. $\square$

5 Quasiprimitive regular maps of type AS and concluding remarks

In this last section, we focus on the fourth type of quasiprimitive actions which admit (orientably-) regular maps of the kind we consider in this paper. This is the *almost simple* **AS** type. A group G is *almost simple* if $T \leq G \leq \mathrm{Aut}(T)$ for some non-Abelian simple group T. Only rudimentary information about regular and orientably-regular maps with an almost simple automorphism group is available, and we sum up the known results in the next few paragraphs.

The most well understood class of almost simple groups from this point of view are the groups $\mathrm{PSL}(2,q)$ and $\mathrm{PGL}(2,q)$, q a prime power. A complete classification of regular and orientably-regular maps of with automorphism group isomorphic to $\mathrm{PSL}(2,q)$ or $\mathrm{PGL}(2,q)$, too complex to be reproduced here, can be extracted from [28] in the orientable case and from [6] for both orientable as well as non-orientable maps.

All the (orientably-) regular maps with (orientation-preserving) automorphism group isomorphic to the *simple* group $\mathrm{PSL}(2,q)$ for $q \geq 4$ exhibit, of course, a quasiprimitive action of the group on vertices. The situation is more complex for $\mathrm{PGL}(2,q)$, $q \geq 4$, and can be sorted out with the help of Propositions 2.1, 2.2, 3.1, 3.2 and 4.5 of [6]. In order to state a simplified version of the result (which can be elaborated on by examining explicit generating sets listed in the above propositions), let θ be the unique automorphism of $GF(q^2)$ of order two and, for any 2×2 matrix u over $GF(q^2)$ representing an element $\pm u \in \mathrm{PGL}(2,q)$, let $\mathrm{tr}(u)$ be the trace of u. Also, we may assume to be working with a 'canonical' copy of $\mathrm{PGL}(2,q)$ within $\mathrm{PSL}(2,q^2)$ consisting of elements $\pm u \in \mathrm{PSL}(2,q^2)$ such that $\mathrm{tr}(\theta u) = \pm u$.

Proposition 5.1 *Let $G = H = \mathrm{PGL}(2,q)$ for a prime power $q \geq 4$. If $\mathcal{M} = (G; x, y, z)$ is a regular map, then G is quasiprimitive on vertices of $\mathcal{M}$ if and only if $\mathrm{tr}(\theta y) = -\mathrm{tr}(y)$ or $\mathrm{tr}(\theta z) = -\mathrm{tr}(z)$. Similarly, if $\mathcal{M} = (H; \lambda, \rho)$ is an orientably-regular map, then H is quasiprimitive on vertices of $\mathcal{M}$ if and only if $\mathrm{tr}(\theta\rho) = -\mathrm{tr}(\rho)$.*

Besides the groups $\mathrm{PGL}(2,q)$, the almost simple 'twisted linear groups' $M(q^2)$ for odd prime powers $q \geq 3$ are the 'other' family of sharply 3-transitive groups for which enumeration of (orientably-) regular maps is known [9]. All the maps supported by the groups $M(q^2)$ turn out to be orientable, and an inspection of the generating sets in [9] quickly shows that *all* orientably-regular maps with orientation preserving automorphism group isomorphic to $M(q^2)$ satisfy the property that the group acts quasiprimitively on their vertex set.

As regards further families of finite non-Abelian simple groups supporting regular maps, classification attempts are available just for the (small) Ree groups and Suzuki groups. In [12] one can find a classification of orientably-regular maps $\mathcal{M}$ of type (ℓ, m) with $\mathrm{Aut}^+(\mathcal{M})$ isomorphic to a Ree group ${}^2G_2(3^n)$ for odd $n > 1$, $\ell = 3$, and $m = 7$, $m = 9$, and for all primes $m \equiv 11 \bmod 12$; all these maps happen to be chiral. A classification of orientably-regular maps $\mathcal{M}$ of type $(4,5)$ with $\mathrm{Aut}^+(\mathcal{M})$ isomorphic to a Suzuki group $Sz(2^n)$ for odd $n > 1$ was given in [13], with all maps chiral again.

Unlike attempts at classifications, the *existence* of at least one (orientably-) regular map with a given *simple* automorphism group is a different matter. A deep study in [22] implies that every finite non-Abelian simple group can be generated by an involution and a non-involution, which means that:

Theorem 5.2 *Every finite non-Abelian simple group is the orientation-preserving automorphism group of an orientably-regular map.*

More specific information about possible types of orientably-regular maps arising from simple groups may be beyond the reach of current methods. Nevertheless, by [19] we know the following:

Theorem 5.3 *Let G be a finite non-Abelian simple group other than $Sp_4(2^n)$, $Sp_4(3^n)$ and ${}^2B_2(2^{2n+1})$. Then, up to a finite number of possible exceptions, G is the automorphism group of an orientably-regular map of type $(3, m)$ for some m.*

A *regular* map with a simple (full) automorphism group must be *non-orientable*, as the automorphism group of a regular map on an orientable surface has a (normal) subgroup of index two. Here, the following result (set up on the basis of a communication with M. Conder [4] and incorporating the recent correction found by M. Mačaj [20]) summarises results of Nuzhin [23, 24, 25] on infinite classes of finite simple groups, Timofeenko [31] on sporadic simple groups, Ashaev (unpublished) on the Baby Monster group and Norton (unpublished) on the Fischer-Griess Monster group.

Theorem 5.4 *A finite simple group is the automorphism group of a regular map if and only if the group is not isomorphic to any of the following: $PSL(3, q)$ and $PSU(3, q)$ for q a prime power, $PSL(4, q)$ and $PSU(4, q)$ for q a power of 2, A_6, A_7, $U_4(3)$, $U_5(2)$, the sporadic Mathieu groups M_{11}, M_{22}, M_{23}, and the McLaughlin group McL.*

Needless to say that the simple (orientation-preserving) automorphism groups of the above (orientably-) regular maps act quasiprimitively on vertex sets of the maps.

Summing up the results obtained in this paper, we derived a full classification of vertex-quasiprimitive regular and orientably-regular maps with automorphism groups of type **HA**, presented new infinite families of such maps with groups of type **TW** and **PA**, outlined the state-of-the-art for maps with groups of type **AS**, and excluded the remaining four types of quasiprimitive groups present in the O'Nan-Scott-Praeger classification. In addition to trying to obtain the classifications for the **TW**, **PA** and **AS** types (which is not going to be an easy task – as can be seen from the **AS** case), in principle, one could think of extending our results to regular and orientably-regular *hypermaps*, which can be identified with group presentations as in (1) and (2) except that the exponent 2 is replaced by any integer ≥ 2, and possibly, to higher-dimensional regular *polytopes*. We leave these as open problems.

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